Agent Models of 1st and 2nd order: from micro to macro

J. A. Carrillo

Imperial College London

Università di Verona, November 2018

Outline

Outline

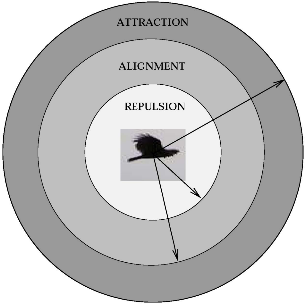
Individual Based Models (Particle models)

Swarming = Aggregation of agents of similar size and body type generally moving in a coordinated way.
Highly developed social organization: insects (locusts, ants, bees ...), fish, birds, micro-organisms,... and artificial robots for unmanned vehicle operation.

Interaction regions between individuals[a]

[a] Aoki, Helmerijk et al., Barbaro, Birnir et al.

- Repulsion Region: R_k.
- Attraction Region: A_k.
- Orientation Region: O_k.



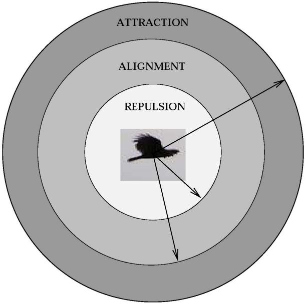
Individual Based Models (Particle models)

Swarming = Aggregation of agents of similar size and body type generally moving in a coordinated way.
Highly developed social organization: insects (locusts, ants, bees ...), fish, birds, micro-organisms,... and artificial robots for unmanned vehicle operation.

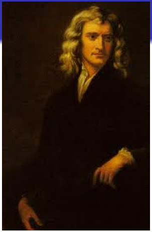
Interaction regions between individuals[a]

[a] Aoki, Helmerijk et al., Barbaro, Birnir et al.

- Repulsion Region: R_k.
- Attraction Region: A_k.
- Orientation Region: O_k.



2nd Order Model: Newton's like equations



D'Orsogna, Bertozzi et al. model (PRL 2006):

$$\begin{cases} \dfrac{dx_i}{dt} = v_i, \\ m\dfrac{dv_i}{dt} = (\alpha - \beta\,|v_i|^2)v_i - \displaystyle\sum_{j\neq i} \nabla K(x_i - x_j). \end{cases}$$

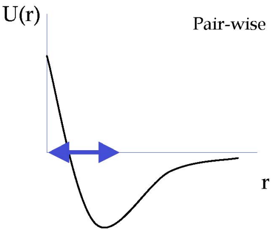
Model assumptions:

- Self-propulsion and friction terms determines an asymptotic speed of $\sqrt{\alpha/\beta}$.
- Attraction/Repulsion modeled by an effective pairwise potential $K(x) = k(r)$.

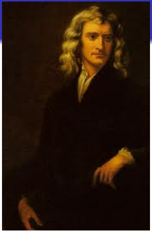
$$k(r) = -C_A e^{-r/\ell_A} + C_R e^{-r/\ell_R}.$$

One can also use Bessel functions in 2D and 3D to produce such a potential.

$C = C_R/C_A > 1$, $\ell = \ell_R/\ell_A < 1$ and $C\ell^2 < 1$:



2nd Order Model: Newton's like equations



D'Orsogna, Bertozzi et al. model (PRL 2006):

$$\begin{cases} \dfrac{dx_i}{dt} = v_i, \\ m\dfrac{dv_i}{dt} = (\alpha - \beta\,|v_i|^2)v_i - \sum_{j \neq i} \nabla K(x_i - x_j). \end{cases}$$

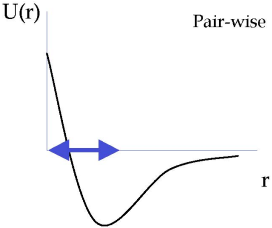
Model assumptions:

- Self-propulsion and friction terms determines an asymptotic speed of $\sqrt{\alpha/\beta}$.
- Attraction/Repulsion modeled by an effective pairwise potential $K(x) = k(r)$.

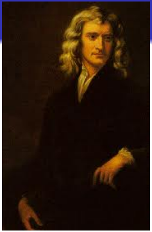
$$k(r) = -C_A e^{-r/\ell_A} + C_R e^{-r/\ell_R}.$$

One can also use Bessel functions in 2D and 3D to produce such a potential.

$C = C_R/C_A > 1$, $\ell = \ell_R/\ell_A < 1$ and $C\ell^2 < 1$:



2nd Order Model: Newton's like equations



D'Orsogna, Bertozzi et al. model (PRL 2006):

$$\begin{cases} \dfrac{dx_i}{dt} = v_i, \\ m\dfrac{dv_i}{dt} = (\alpha - \beta\,|v_i|^2)v_i - \displaystyle\sum_{j\neq i} \nabla K(x_i - x_j). \end{cases}$$

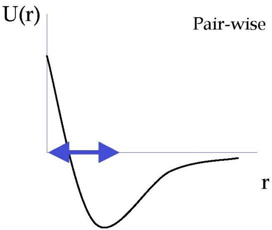
Model assumptions:

- Self-propulsion and friction terms determines an asymptotic speed of $\sqrt{\alpha/\beta}$.
- Attraction/Repulsion modeled by an effective pairwise potential $K(x) = k(r)$.

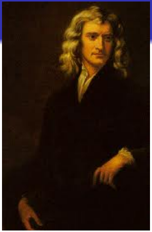
$$k(r) = -C_A e^{-r/\ell_A} + C_R e^{-r/\ell_R}.$$

One can also use Bessel functions in 2D and 3D to produce such a potential.

$C = C_R/C_A > 1$, $\ell = \ell_R/\ell_A < 1$ and $C\ell^2 < 1$:



2nd Order Model: Newton's like equations



D'Orsogna, Bertozzi et al. model (PRL 2006):

$$\begin{cases} \dfrac{dx_i}{dt} = v_i, \\ m\dfrac{dv_i}{dt} = (\alpha - \beta\,|v_i|^2)v_i - \displaystyle\sum_{j\neq i} \nabla K(x_i - x_j). \end{cases}$$

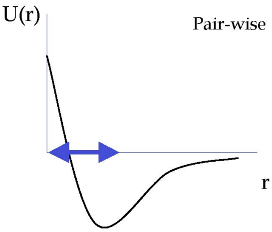
Model assumptions:

- Self-propulsion and friction terms determines an asymptotic speed of $\sqrt{\alpha/\beta}$.
- Attraction/Repulsion modeled by an effective pairwise potential $K(x) = k(r)$.

$$k(r) = -C_A e^{-r/\ell_A} + C_R e^{-r/\ell_R}.$$

One can also use Bessel functions in 2D and 3D to produce such a potential.

$C = C_R/C_A > 1$, $\ell = \ell_R/\ell_A < 1$ and $C\ell^2 < 1$:



Velocity consensus model

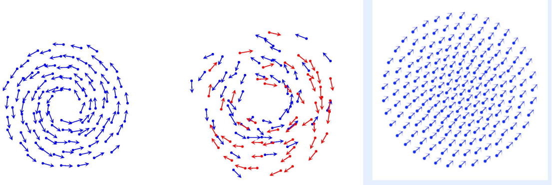
Cucker-Smale Model (IEEE Automatic Control 2007):

$$\begin{cases} \dfrac{dx_i}{dt} = v_i, \\ \dfrac{dv_i}{dt} = \displaystyle\sum_{j=1}^{N} \psi_{ij} \left(v_j - v_i \right), \end{cases}$$

with the communication rate, $\gamma \geq 0$:

$$\psi_{ij} = \psi(|x_i - x_j|) = \frac{1}{\left(1 + |x_i - x_j|^2\right)^{\gamma}}.$$

Typical patterns: milling, double milling or flocking:



Outline

1. Modelling & Levels of Description
   - Collective Behavior Models
   - From micro to macro: PDE models
   - Qualitative Properties & Hydrodynamics
2. Critical thresholds
   - Main equations
   - Euler-Alignment system
   - Euler-Alignment-Poisson system
3. Repulsive Newtonian with Quadractic confinement
   - Main equations
   - Proof
4. Conclusions

Convergence of the particle method

Empirical measures: if $x_i, v_i : [0,T) \to \mathbb{R}^d$, for $i = 1, \ldots, N$, is a solution to the ODE system,

$$\begin{cases} \dfrac{dx_i}{dt} = v_i, \\ \dfrac{dv_i}{dt} = \overbrace{(\alpha - \beta\,|v_i|^2)v_i}^{\text{propulsion-friction}} - \overbrace{\sum_{j \neq i} m_j \nabla K(x_i - x_j)}^{\text{attraction-repulsion}} + \overbrace{\sum_{j=1}^{N} m_j a_{ij}\,(v_j - v_i)}^{\text{orientation}}. \end{cases}$$

then the $f_N : [0,T) \to \mathcal{P}_1(\mathbb{R}^d)$ given by

$$f_N(t) := \sum_{i=1}^{N} m_i \delta_{(x_i(t), v_i(t))} \qquad \text{with} \sum_{i=1}^{N} m_i = 1\,,$$

is expected to be the solution corresponding to initial atomic measures.

Convergence of the particle method

Empirical measures: if $x_i, v_i : [0, T) \to \mathbb{R}^d$, for $i = 1, \dots, N$, is a solution to the ODE system,

$$\begin{cases} \dfrac{dx_i}{dt} = v_i, \\ \dfrac{dv_i}{dt} = \overbrace{(\alpha - \beta\,|v_i|^2)v_i}^{\text{propulsion-friction}} - \overbrace{\sum_{j \neq i} m_j \nabla K(x_i - x_j)}^{\text{attraction-repulsion}} + \overbrace{\sum_{j=1}^{N} m_j a_{ij}\,(v_j - v_i)}^{\text{orientation}}. \end{cases}$$

then the $f_N : [0, T) \to \mathcal{P}_1(\mathbb{R}^d)$ given by

$$f_N(t) := \sum_{i=1}^{N} m_i \delta_{(x_i(t), v_i(t))} \qquad \text{with} \sum_{i=1}^{N} m_i = 1\,,$$

is expected to be the solution corresponding to initial atomic measures.

Convergence of the particle method

Empirical measures: if $x_i, v_i : [0,T) \to \mathbb{R}^d$, for $i = 1, \dots, N$, is a solution to the ODE system,

$$\begin{cases} \dfrac{dx_i}{dt} = v_i, \\ \dfrac{dv_i}{dt} = \overbrace{(\alpha - \beta\,|v_i|^2)v_i}^{\text{propulsion-friction}} - \overbrace{\sum_{j\neq i} m_j \nabla K(x_i - x_j)}^{\text{attraction-repulsion}} + \overbrace{\sum_{j=1}^{N} m_j a_{ij}\,(v_j - v_i)}^{\text{orientation}}. \end{cases}$$

then the $f_N : [0,T) \to \mathcal{P}_1(\mathbb{R}^d)$ given by

$$f_N(t) := \sum_{i=1}^{N} m_i \delta_{(x_i(t),v_i(t))} \qquad \text{with } \sum_{i=1}^{N} m_i = 1\,,$$

is expected to be the solution corresponding to initial atomic measures.

Convergence of the particle method

Empirical measures: if $x_i, v_i : [0,T) \to \mathbb{R}^d$, for $i = 1,\dots,N$, is a solution to the ODE system,

$$\begin{cases} \dfrac{dx_i}{dt} = v_i, \\ \dfrac{dv_i}{dt} = \overbrace{(\alpha - \beta\,|v_i|^2)v_i}^{\text{propulsion-friction}} - \overbrace{\sum_{j\neq i} m_j \nabla K(x_i - x_j)}^{\text{attraction-repulsion}} + \overbrace{\sum_{j=1}^N m_j a_{ij}\,(v_j - v_i)}^{\text{orientation}}. \end{cases}$$

then the $f_N : [0,T) \to \mathcal{P}_1(\mathbb{R}^d)$ given by

$$f_N(t) := \sum_{i=1}^N m_i \delta_{(x_i(t), v_i(t))} \qquad \text{with } \sum_{i=1}^N m_i = 1\,,$$

is expected to be the solution corresponding to initial atomic measures.

Mesoscopic models

Model with asymptotic velocity + Attraction/Repulsion:

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f + \mathrm{div}_v[(\alpha - \beta|v|^2)vf] - \mathrm{div}_v\left[(\nabla_x K \star \rho)f\right] = 0.$$

Velocity consensus Model:

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = \nabla_v \cdot \left[\underbrace{\left(\int_{\mathbb{R}^{2d}} \frac{v-w}{(1+|x-y|^2)^\gamma} f(y,w,t)\, dy\, dw \right)}_{:=\xi(f)(x,v,t)} f(x,v,t) \right]$$

Orientation, Attraction and Repulsion:

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f - \mathrm{div}_v\left[(\nabla_x K \star \rho)f\right] = \nabla_v \cdot \left[\xi(f)(x,v,t) f(x,v,t)\right].$$

Mesoscopic models

Model with asymptotic velocity + Attraction/Repulsion:

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f + \mathrm{div}_v[(\alpha - \beta|v|^2)vf] - \mathrm{div}_v\left[(\nabla_x K \star \rho)f\right] = 0.$$

Velocity consensus Model:

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = \nabla_v \cdot \left[\underbrace{\left(\int_{\mathbb{R}^{2d}} \frac{v - w}{(1 + |x-y|^2)^\gamma} f(y,w,t)\, dy\, dw \right)}_{:=\xi(f)(x,v,t)} f(x,v,t) \right]$$

Orientation, Attraction and Repulsion:

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f - \mathrm{div}_v\left[(\nabla_x K \star \rho)f\right] = \nabla_v \cdot \left[\xi(f)(x,v,t) f(x,v,t)\right].$$

Mesoscopic models

Model with asymptotic velocity + Attraction/Repulsion:

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f + \mathrm{div}_v[(\alpha - \beta |v|^2) v f] - \mathrm{div}_v\left[(\nabla_x K \star \rho) f\right] = 0.$$

Velocity consensus Model:

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = \nabla_v \cdot \left[\underbrace{\left(\int_{\mathbb{R}^{2d}} \frac{v - w}{(1 + |x - y|^2)^\gamma} f(y, w, t)\, dy\, dw \right)}_{:= \xi(f)(x,v,t)} f(x, v, t) \right]$$

Orientation, Attraction and Repulsion:

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f - \mathrm{div}_v\left[(\nabla_x K \star \rho) f\right] = \nabla_v \cdot \left[\xi(f)(x, v, t) f(x, v, t)\right].$$

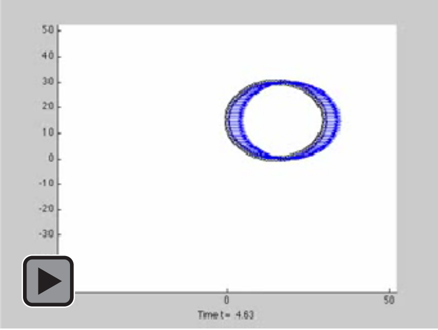
Outline

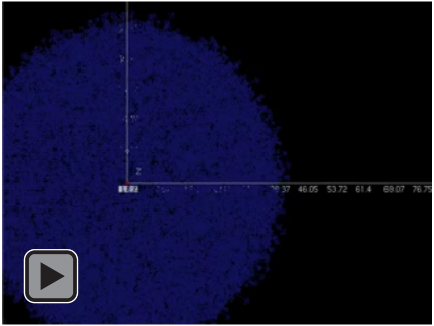
Macroscopic equations

Monokinetic Solutions

Assuming that there is a deterministic velocity for each position and time, $f(x,v,t) = \rho(x,t)\,\delta(v - u(x,t))$ is a distributional solution if and only if,

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \mathrm{div}_x(\rho u) = 0, \\[2ex] \rho\,\dfrac{\partial u}{\partial t} + \rho\,(u\cdot\nabla_x)u = \rho\,(\alpha - \beta|u|^2)u - \rho\,(\nabla_x K \star \rho). \end{cases}$$





Outline

Main equations

Euler equations with nonlocal forces(alignment-attractive/repulsive forces):

$$\partial_t \rho + \partial_x(\rho u) = 0, \quad x \in \mathbb{R}, \quad t \geq 0,$$
$$\partial_t u + u\partial_x u = \int_{\mathbb{R}} \psi(x-y)(u(y) - u(x))\rho(y)\, dy - \partial_x K \star \rho,$$

Basic assumptions:

- ρ is a probability density function, i.e., $\|\rho(\cdot,t)\|_{L^1} = 1$.
- The influence function $\psi \in W^{1,\infty}(\mathbb{R})$ is symmetry and uniformly bounded:

$$0 \leq \psi_m \leq \psi(x) = \psi(-x) \leq \psi_M.$$

Outline

Euler-Alignment system

We consider the Euler-Alignment system:

$$
\begin{aligned}
&\partial_t \rho + \partial_x(\rho u) = 0, \quad x \in \mathbb{R}, \quad t \geq 0, \\
&\partial_t u + u\partial_x u = \int_{\mathbb{R}} \psi(x-y)(u(y)-u(x))\rho(y)\, dy.
\end{aligned}
$$

Idea of the proof: Differentiate the velocity equation with respect to x to get

$$
(\partial_t + u\partial_x)v = -v^2 - (\psi \star \rho)v + \partial_x\psi \star (\rho u) - u\partial_x(\psi \star \rho),
$$

where $v = \partial_x u$.

Goal: Classify the initial configurations that leading to global regularity or finite time blow-up of solutions:

- If $v_0 > \sigma_+$, $v(t)$ exists for all time.
- If $v_0 < \sigma_-$, $v(t) \to -\infty$ in finite time.

Previous result

Tadmor-Tan(Proc. Royal Soc. A, 2014):

$$(\partial_t + u\partial_x)v = \underbrace{-v^2}_{Bad} \underbrace{-(\psi \star \rho)v}_{Good} \underbrace{+\partial_x\psi \star (\rho u) - u\partial_x(\psi \star \rho)}_{Bad}.$$

Main idea: Compact support of the density ρ & Large-time behaviour

$$S(t) := \sup_{x,y \in \text{supp}(\rho(t))} |x-y| \le D < \infty,$$

$$V(t) := \sup_{x,y \in \text{supp}(\rho(t))} |u(x,t) - u(y,t)| \to 0 \text{ as } t \to \infty,$$

exponentially fast.

We now know how to hand the "Good" and "Bad" terms.

- $\psi \star \rho \ge \psi(D) > 0$
- $\|\partial_x\psi \star (\rho u) - u\partial_x(\psi \star \rho)\|_{L^\infty} \lesssim e^{-Ct}$

Previous result

Set

$$v_0 := \inf_{x \in \text{supp}(\rho_0)} \partial_x u_0(x) \quad \text{and} \quad V_0 := \sup_{x,y \in \text{supp}(\rho_0)} |u_0(x) - u_0(y)|.$$

Theorem (Tadmor-Tan, 2014)

• *(Subcritical region)If the initial configurations satisfy*

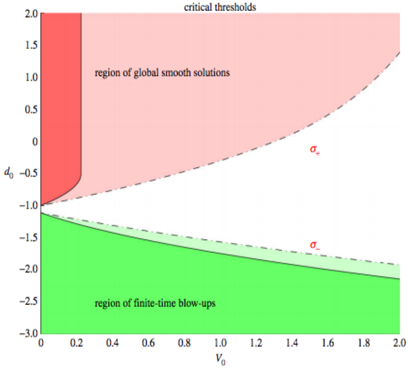
$$V_0 \leq \frac{\psi^2(D)m}{4\|\psi\|_{\dot{W}^{1,\infty}}} \quad \textit{and} \quad v_0 \geq -\frac{1}{2}\left(\psi(D) + \sqrt{\psi^2(D) - 4V_0\|\psi\|_{\dot{W}^{1,\infty}}}\right),$$

then $\partial_x u(x,t)$ remains uniformly bounded for all $(x,t) \in supp(\rho)$.

• *(Supercritical region)If $v_0 < -\frac{1}{2}\left(1 + \sqrt{1 + 4V_0\|\psi\|_{\dot{W}^{1,\infty}}}\right)$, then there exists a finite time T_c such that*

$$\inf_{x \in supp(\rho(\cdot,t))} \partial_x u(x,t) \to -\infty \quad as \quad t \to T_c - .$$

Previous result



Weakness:

- The results are not sharp, in fact, $\sigma_+ \geq \psi \star \rho \geq \sigma_-$.
- The estimate of large-time behavior is essential, that is, if we can not obtain the large-time behavior of solutions, there is nothing we can do.

New idea of the proof

C.-Choi-Tadmor-Tan (M3AS, 2016):

$$(\partial_t + u\partial_x)v = \underbrace{-v^2}_{Bad} \underbrace{-(\psi \star \rho)v}_{Good} \underbrace{+\partial_x\psi \star (\rho u) - u\partial_x(\psi \star \rho)}_{Not\ that\ bad}.$$

It follows from the symmetry of the influence function ψ that

$$\partial_x\psi \star (\rho u) = -\psi \star \partial_t\rho.$$

This yields that

$$(\partial_t + u\partial_x)v \underbrace{+\partial_t(\psi \star \rho) + u\partial_x(\psi \star \rho)}_{Not\ that\ bad} = \underbrace{-v^2}_{Bad} \underbrace{-(\psi \star \rho)v}_{Good},$$

and

$$(v + \psi \star \rho)' = -v(v + \psi \star \rho),$$

where $'$ denotes the time derivative along the characteristic flow.

We now set $d := v + \psi \star \rho$. Then we find

$$\rho' = -\rho(d - \psi \star \rho),$$
$$d' = -d(d - \psi \star \rho).$$

Proposition:

- If $d_0 < 0$, $d \to -\infty$ in finite time.
- If $d_0 = 0$, $d(t) = 0$ for all $t \geq 0$.
- If $d_0 > 0$, $d(t) \to \psi \star \rho$ as $t \to \infty$.

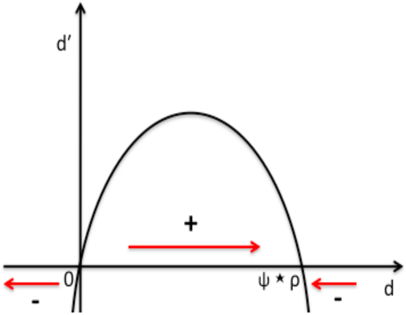


Figure: d vs d'

Theorem

Consider the Euler-Alignment system.
• *(Subcritical region) If* $\partial_x u_0(x) \geq -\psi \star \rho_0(x)$ *for all* $x \in \mathbb{R}$*, the system has a global classical solution, namely,*

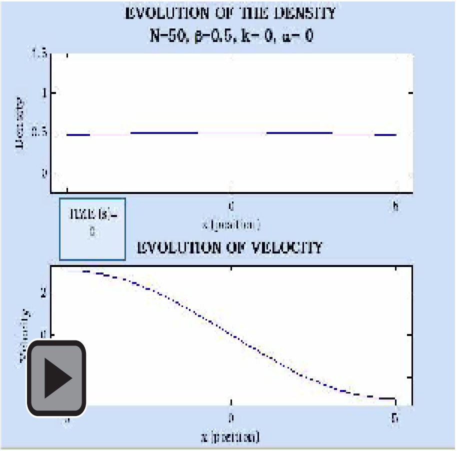
$$(\rho, u) \in \mathcal{C}(\mathbb{R}^+; L^\infty(\mathbb{R})) \times \mathcal{C}(\mathbb{R}^+; \dot{W}^{1,\infty}(\mathbb{R})).$$

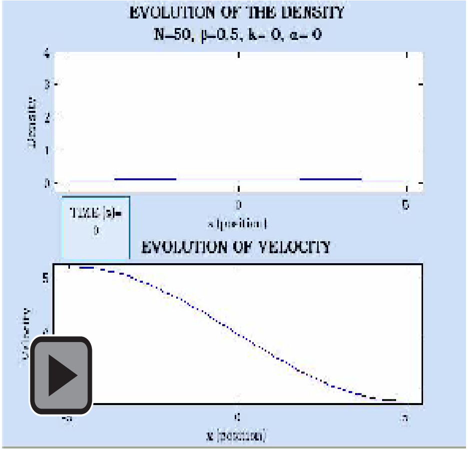
• *(Supercritical region) If there exists an* x *such that* $\partial_x u_0(x) < -\psi \star \rho_0(x)$*, the solution blows up in a finite time.*

Strength:

- Complete description of critical thresholds; No gap between two thresholds.
- Compactly supported initial density is not required, and furthermore, we do not need to have the estimate of large-time behavior of solutions.

Numerical illustration by Lagrangian Methods[1]





[1] C.-Choi-Pérez, book chapter edited by Bellomo, Degond & Tadmor

Outline

Euler-Alignment-Poisson system

Consider Euler-Alignment-Poisson system:

$$\partial_t \rho + \partial_x(\rho u) = 0,$$
$$\partial_t u + u\partial_x u = -k\partial_x \phi + \int_{\mathbb{R}} \psi(x-y)(u(y,t) - u(x,t))\rho(y,t)dy,$$
$$\partial_x^2 \phi = \rho.$$

• $k > 0$; attractive, $k < 0$; repulsive

Similarly, we find

$$\rho' = -\rho(d - \psi \star \rho),$$
$$d' = -d(d - \psi \star \rho) + k\rho.$$

Set $\beta = d/\rho$, then we obtain

$$\beta' = -k, \quad \text{i.e.,} \quad \beta(t) = \beta_0 - kt.$$

Attractive Poisson forcing($k > 0$)

Using the estimate of β, we get

$$\rho' = -\rho(d - \psi \star \rho) = -\rho(\rho(\beta_0 - kt) - \psi \star \rho) = -\beta_0 \rho^2 + kt\rho^2 + \rho(\psi \star \rho).$$

Then we obtain the explicit form of solution ρ:

$$\rho^{-1}(t) = e^{-\int_0^t (\psi \star \rho) ds} \left(\rho_0^{-1} + \int_0^t (\beta_0 - ks) e^{\int_0^s (\psi \star \rho) d\tau} ds \right).$$

For the attractive case $k > 0$, $\beta_0 - ks$ becomes negative in finite time, irrespective of the value of β_0.

If $k > 0$, $\rho(t) \to +\infty$ in finite time.

• In the attractive case, the blowup is "unconditional", independent of the choice of initial configurations. This indicates that Poisson force dominates the alignment force.

Repulsive Poisson forcing $k < 0$

Notice that if $\beta_0 \geq 0$, then we can easily find that $\rho(t)$ remains bounded for all $t \geq 0$ due to $\beta \geq 0$. It exactly gives us the same subcritical region with the one of Euler-Alignment system.

Consider the case when $\beta_0 < 0$. Since $\beta_0 - ks < 0$ for $s \leq \beta_0/k$, we obtain

$$\rho^{-1}(t) = \rho_0^{-1} + \underbrace{\int_0^{\frac{\beta_0}{k}} (\beta_0 - ks) e^{\int_0^s (\psi \star \rho) d\tau} ds}_{Negative} + \underbrace{\int_{\frac{\beta_0}{k}}^t (\beta_0 - ks) e^{\int_0^s (\psi \star \rho) d\tau} ds}_{Positive}$$

$$\rho(\cdot, t) \text{ remains bounded} \quad \Longleftrightarrow \quad \rho_0^{-1} + \int_0^{\frac{\beta_0}{k}} (\beta_0 - ks) e^{\int_0^s (\psi \star \rho) d\tau} ds > 0.$$

Repulsive Poisson forcing $k < 0$

Theorem

- *(Subcritical region) If* $\partial_x u_0(x) > -\psi \star \rho_0(x) + \sigma_+(x)$ *for all* $x \in \mathbb{R}$, *then the system has a global classical solution. Here,* $\sigma_+(x) = 0$ *whenever* $\rho_0(x) = 0$ *and elsewhere* $\sigma_+(x)$ *is the (unique) negative root of the equation*

$$\rho_0^{-1}(x) - \frac{1}{\psi_M^2}\left(k + \psi_M \sigma_+(x)/\rho_0(x) - k e^{\psi_M \sigma_+(x)/k\rho_0(x)}\right) = 0.$$

- *(Supercritical region) If there exists an* x *such that*

$$\partial_x u_0(x) < -\psi \star \rho_0(x) + \sigma_-(x), \quad \sigma_-(x) := -\sqrt{-2k\rho_0(x)},$$

then the solution blows up in a finite time.

Repulsive Poisson forcing $k < 0$

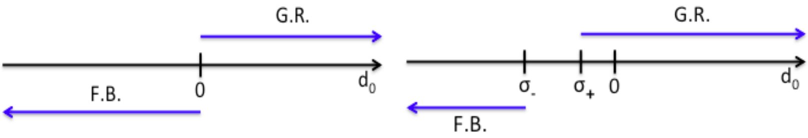
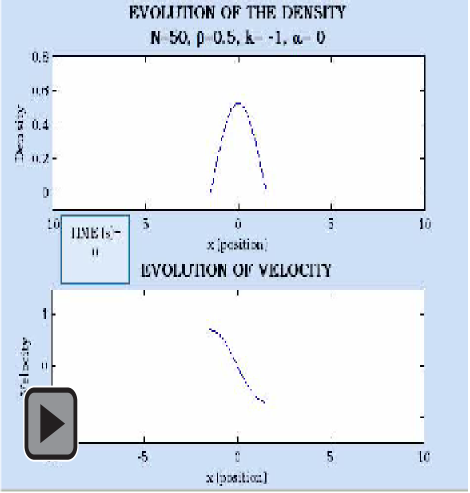
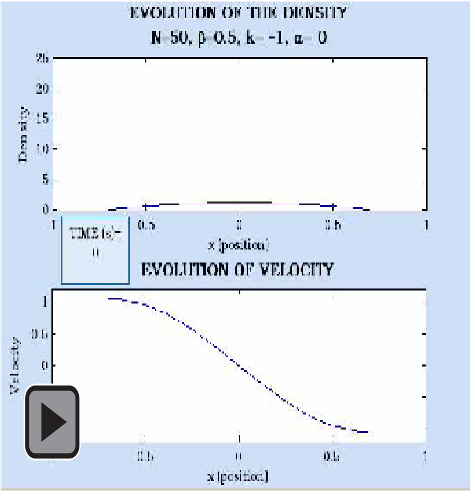


Figure: Euler-Alignment system vs Euler-Alignment-Poisson system

• The repulsive force enhances regularity. Indeed, we have a larger subcritical region than the case of $K \equiv 0$.

Numerical illustration by Lagrangian Methods[2]





[2]C.-Choi-Pérez, book chapter edited by Bellomo, Degond & Tadmor

Outline

Main equations

Euler equations with Newtonian repulsion and quadratic confinement:

$$\partial_t \rho + \partial_x(\rho u) = 0, \quad x \in \mathbb{R}, \quad t \geq 0,$$
$$\partial_t(\rho u) + \partial_x(\rho u^2) = -\rho u - \rho \partial_x K \star \rho,$$

where $-|x| + \frac{x^2}{2}$. [3]

Initial data: density compactly supported in $\Omega_0 := \Omega(0) = (a_0, b_0)$ with

$$(\rho(t,\cdot), u(t,\cdot))|_{t=0} = (\rho_0, u_0) \in H^2(\Omega_0) \times H^3(\Omega_0),$$

The initial mass and momentum are:

$$0 < M_0 := \int_{\Omega_0} \rho_0(x) dx \qquad \text{and} \qquad M_1 := \int_{\Omega_0} \rho_0(x) u_0(x)\, dx\,.$$

Lagrangian solutions: $f(t,x) := \rho(t, \eta(t,x))$ and $v(t,x) := u(t, \eta(t,x))$ with

$$\frac{d\eta(t,x)}{dt} = u(t, \eta(t,x)) \quad \text{with} \quad \eta(0,x) = x \in \Omega_0\,.$$

[3] see also S. Engelberg, H. Liu and E. Tadmor (Indiana Univ. Math. J. 2001) for critical thresholds.

Main equations

Euler equations with Newtonian repulsion and quadratic confinement:

$$\partial_t \rho + \partial_x(\rho u) = 0, \quad x \in \mathbb{R}, \quad t \geq 0,$$
$$\partial_t(\rho u) + \partial_x(\rho u^2) = -\rho u - \rho \partial_x K \star \rho,$$

where $-|x| + \frac{x^2}{2}$. [3]

Initial data: density compactly supported in $\Omega_0 := \Omega(0) = (a_0, b_0)$ with

$$(\rho(t,\cdot), u(t,\cdot))|_{t=0} = (\rho_0, u_0) \in H^2(\Omega_0) \times H^3(\Omega_0),$$

The initial mass and momentum are:

$$0 < M_0 := \int_{\Omega_0} \rho_0(x) dx \qquad \text{and} \qquad M_1 := \int_{\Omega_0} \rho_0(x) u_0(x)\, dx\,.$$

Lagrangian solutions: $f(t,x) := \rho(t,\eta(t,x))$ and $v(t,x) := u(t,\eta(t,x))$ with

$$\frac{d\eta(t,x)}{dt} = u(t,\eta(t,x)) \quad \text{with} \quad \eta(0,x) = x \in \Omega_0\,.$$

[3] see also S. Engelberg, H. Liu and E. Tadmor (Indiana Univ. Math. J. 2001) for critical thresholds.

Main equations

Euler equations with Newtonian repulsion and quadratic confinement:

$$\partial_t \rho + \partial_x(\rho u) = 0, \quad x \in \mathbb{R}, \quad t \geq 0,$$
$$\partial_t(\rho u) + \partial_x(\rho u^2) = -\rho u - \rho \partial_x K \star \rho,$$

where $-|x| + \frac{x^2}{2}$. [3]

Initial data: density compactly supported in $\Omega_0 := \Omega(0) = (a_0, b_0)$ with

$$(\rho(t,\cdot), u(t,\cdot))|_{t=0} = (\rho_0, u_0) \in H^2(\Omega_0) \times H^3(\Omega_0),$$

The initial mass and momentum are:

$$0 < M_0 := \int_{\Omega_0} \rho_0(x) dx \qquad \text{and} \qquad M_1 := \int_{\Omega_0} \rho_0(x) u_0(x)\, dx\,.$$

Lagrangian solutions: $f(t,x) := \rho(t, \eta(t,x))$ and $v(t,x) := u(t, \eta(t,x))$ with

$$\frac{d\eta(t,x)}{dt} = u(t, \eta(t,x)) \quad \text{with} \quad \eta(0,x) = x \in \Omega_0\,.$$

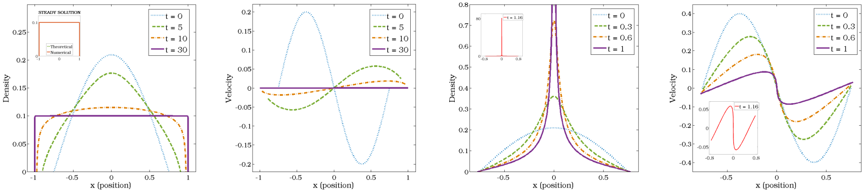
[3] see also S. Engelberg, H. Liu and E. Tadmor (Indiana Univ. Math. J. 2001) for critical thresholds.

Stationary States & Numerical Simulation

$$\rho_\infty(x) = \frac{M_0}{2} \quad \text{and} \quad u_\infty(x) = 0 \quad \text{for} \quad x \in \Omega_\infty := (\Gamma - 1, \Gamma + 1)$$

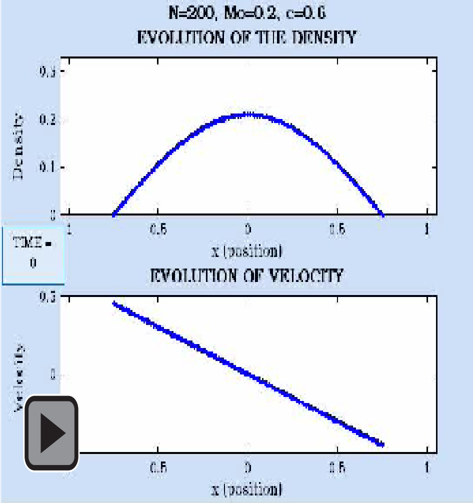
with

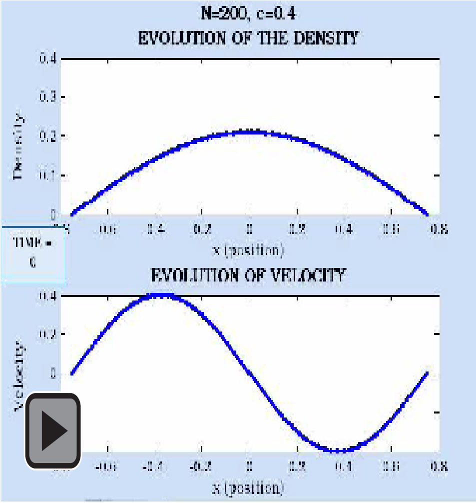
$$\Gamma := \frac{1}{M_0}\left(\int_{\mathbb{R}} x\rho_0(x)\,dx + \int_{\mathbb{R}} \rho_0(x)u_0(x)\,dx\right).$$



(a) (b) (c) (d)

Numerical illustration by Lagrangian Methods[4]





[4] C.-Choi-Pérez, book chapter edited by Bellomo, Degond & Tadmor

Main Result[5]

Blow-up versus global existence

Assume that (f, v) is a classical solution to the hydrodynamic system, then:
Case A: If $1 - 4M_0 > 0$, the solution blows up in finite time if and only if there exists a $x^* \in \Omega_0$ such that

$$\partial_x u_0(x^*) < 0, \quad M_0 - 2\rho_0(x^*) < \lambda_1 \partial_x u_0(x^*),$$

and

$$2\rho_0(x^*) \leq (\lambda_1 \partial_x u_0(x^*) - M_0 + 2\rho_0(x^*))^{-\lambda_2/\sqrt{\Xi}} (\lambda_2 \partial_x u_0(x^*) - M_0 + 2\rho_0(x^*))^{\lambda_1/\sqrt{\Xi}}.$$

Case B: If $1 - 4M_0 = 0$, the solution blows up in finite time if and only if there exists a $x^* \in \Omega_0$ such that

$$\partial_x u_0(x^*) < \min\left\{0, 4\rho_0(x^*) - \frac{1}{2}\right\},$$

and
$$\log\left(\frac{8\rho_0(x^*)}{8\rho_0(x^*) - 2\partial_x u_0(x^*) - 1}\right) \leq \frac{2\partial_x u_0(x^*)}{8\rho_0(x^*) - 2\partial_x u_0(x^*) - 1}.$$

Case C: If $1 - 4M_0 < 0$: more complicated conditions but an if and only if.

[5] C.-Choi-Zatorska, M3AS 2016

Main Result[6]

Asymptotic Behavior

Moreover, for all cases, if there is no finite-time blow-up, then the classical solution (f, v) exists globally in time and it satisfies

$$f_\infty(x) := \lim_{t\to\infty} f(t,x) = \frac{M_0}{2} \quad \text{and} \quad v_\infty(x) := \lim_{t\to\infty} v(t,x) = 0 \quad \text{for all } x \in \Omega_0,$$

exponentially fast. Moreover, the characteristic flow satisfies

$$\eta_\infty(x) := \lim_{t\to\infty} \eta(t,x) = \frac{1}{M_0}\left(\int_{\Omega_0} y\rho_0(y)\,dy + \int_{\Omega_0} \rho_0(y)\,u_0(y)\,dy + 2\int_{a_0}^{x} \rho_0(y)\,dy - M_0\right)$$

for all $x \in \Omega_0$. In particular, $\Omega(t) = (a(t), b(t))$ and

$$\lim_{t\to\infty} |a(t) - \Gamma + 1| = 0 \quad \text{and} \quad \lim_{t\to\infty} |b(t) - \Gamma - 1| = 0\,,$$

exponentially fast. As a consequence, there exists $C > 0$ depending on the L^∞ bounds of ρ_0 and $\partial_x u_0$ in Ω_0 and $\lambda > 0$ depending on the initial mass M_0 such that

$$\|\rho(t,\cdot) - \rho_\infty(\cdot)\|_{L^1(\mathbb{R})} \le Ce^{-\lambda t}\,.$$

Outline

Main Ideas

- Using the characteristic flow, it is easy to check that (ρ, u) is a local-in-time classical solution of the pressure-less Euler-type system if and only if (f, v) is a classical solution of the system

$$
\begin{aligned}
f(t,x)\frac{\partial \eta(t,x)}{\partial x} &= \rho_0(x), \\
\partial_t v(t,x) + v(t,x) &= -\int_{\Omega(t)} W'(\eta(t,x) - y)\rho(t,y)dy \\
&= -\int_{\Omega_0} W'(\eta(t,x) - \eta(t,y))\rho_0(y)\, dy,
\end{aligned}
$$

for $(t,x) \in (0,\infty) \times \Omega_0$, where we used the conservation of mass.

- Taking a further t-derivative on the second equation, we deduce

$$
\begin{aligned}
\partial^2_{tt} v(t,x) + \partial_t v(t,x) &= -\int_{\Omega_0} \partial^2 W(\eta(t,x) - \eta(t,y))\,(v(t,x) - v(t,y))\,\rho_0(y)\, dy \\
&= -vM_0 + \int_{\Omega_0} v(t,y)\rho_0(y)\, dy.
\end{aligned}
$$

Main Ideas

- Using the characteristic flow, it is easy to check that (ρ, u) is a local-in-time classical solution of the pressure-less Euler-type system if and only if (f, v) is a classical solution of the system

$$f(t,x)\frac{\partial \eta(t,x)}{\partial x} = \rho_0(x),$$

$$\partial_t v(t,x) + v(t,x) = -\int_{\Omega(t)} W'(\eta(t,x) - y)\rho(t,y)dy$$

$$= -\int_{\Omega_0} W'(\eta(t,x) - \eta(t,y))\rho_0(y)\, dy,$$

for $(t,x) \in (0,\infty) \times \Omega_0$, where we used the conservation of mass.

- Taking a further t-derivative on the second equation, we deduce

$$\partial_{tt}^2 v(t,x) + \partial_t v(t,x) = -\int_{\Omega_0} \partial^2 W(\eta(t,x) - \eta(t,y))\,(v(t,x) - v(t,y))\,\rho_0(y)\, dy$$

$$= -vM_0 + \int_{\Omega_0} v(t,y)\rho_0(y)\, dy.$$

Main Ideas

- Evolution of the first moment:

$$\int_{\Omega_0} v(t,x)\rho_0(x)\,dx = e^{-t}\int_{\Omega_0} \rho_0(x)u_0(x)\,dx.$$

This leads to an explicit second order ODE for the velocity field over characteristics: $\partial_{tt}^2 v + \partial_t v + M_0 v = M_1 e^{-t}$ for $t > 0$.

- Solving explicitly the ODE for v leads to explicit formulas for both η and $\partial_x\eta$. Blow-up happens if and only if there exists $t_* > 0$ and $x_* \in \Omega_0$ such that $\partial_x\eta(t_*, x_*) = 0$. The first theorem is proved after careful study of the different cases for the ODE.

- The second theorem is shown by carefully estimating the difference between the solution and an intermediate profile given by

$$\bar{\rho}(t,y) = \frac{M_0}{|\Omega(t)|}\chi_{\Omega(t)}(y) \text{ for } y \in \mathbb{R}.$$

Main Ideas

- Evolution of the first moment:

$$\int_{\Omega_0} v(t,x)\rho_0(x)\,dx = e^{-t}\int_{\Omega_0} \rho_0(x)u_0(x)\,dx.$$

  This leads to an explicit second order ODE for the velocity field over characteristics: $\partial^2_{tt}v + \partial_t v + M_0 v = M_1 e^{-t}$ for $t > 0$.

- Solving explicitly the ODE for v leads to explicit formulas for both η and $\partial_x\eta$. Blow-up happens if and only if there exists $t_* > 0$ and $x_* \in \Omega_0$ such that $\partial_x\eta(t_*, x_*) = 0$. The first theorem is proved after careful study of the different cases for the ODE.

- The second theorem is shown by carefully estimating the difference between the solution and an intermediate profile given by

$$\bar\rho(t,y) = \frac{M_0}{|\Omega(t)|}\chi_{\Omega(t)}(y) \text{ for } y \in \mathbb{R}.$$

Main Ideas

- Evolution of the first moment:

$$\int_{\Omega_0} v(t,x)\rho_0(x)\,dx = e^{-t}\int_{\Omega_0} \rho_0(x)u_0(x)\,dx.$$

This leads to an explicit second order ODE for the velocity field over characteristics: $\partial^2_{tt}v + \partial_t v + M_0 v = M_1 e^{-t}$ for $t > 0$.

- Solving explicitly the ODE for v leads to explicit formulas for both η and $\partial_x\eta$. Blow-up happens if and only if there exists $t_* > 0$ and $x_* \in \Omega_0$ such that $\partial_x\eta(t_*, x_*) = 0$. The first theorem is proved after careful study of the different cases for the ODE.

- The second theorem is shown by carefully estimating the difference between the solution and an intermediate profile given by

$$\bar\rho(t,y) = \frac{M_0}{|\Omega(t)|}\chi_{\Omega(t)}(y) \text{ for } y \in \mathbb{R}\,.$$

Conclusions & Open Problems

- Simple modelling of the three main mechanisms leads to complicated patterns.
- Hydrodynamic Equations without pressure but with nonlocal terms can be at least formally derived.
- Critical thresholds in 1D are obtained for the Euler-type equations. Sharp criteria for alignment but not with attractive-repulsive potentials.
- Sharp Results for thresholds and asymptotic behavior for teh particular case of Newtonian repulsive confined quadratically.
- References:

1. C.-D'Orsogna-Panferov (KRM 2008).
2. C.-Fornasier-Toscani-Vecil (Birkhäuser 2011)
3. C.-Klar-Martin-Tiwari (M3AS 2010).
4. C.-Choi-Tadmor-Tan (M3AS 2016)
5. C.-Choi-Zatorska (M3AS 2016)
6. C.-Choi-Pérez (Book Chapter, Birkhäuser volume edited by Bellomo, Degond & Tadmor)

Conclusions & Open Problems

- Simple modelling of the three main mechanisms leads to complicated patterns.
- Hydrodynamic Equations without pressure but with nonlocal terms can be at least formally derived.
- Critical thresholds in 1D are obtained for the Euler-type equations. Sharp criteria for alignment but not with attractive-repulsive potentials.
- Sharp Results for thresholds and asymptotic behavior for teh particular case of Newtonian repulsive confined quadratically.
- References:

1. C.-D'Orsogna-Panferov (KRM 2008).
2. C.-Fornasier-Toscani-Vecil (Birkhäuser 2011)
3. C.-Klar-Martin-Tiwari (M3AS 2010).
4. C.-Choi-Tadmor-Tan (M3AS 2016)
5. C.-Choi-Zatorska (M3AS 2016)
6. C.-Choi-Pérez (Book Chapter, Birkhäuser volume edited by Bellomo, Degond & Tadmor)

Conclusions & Open Problems

- Simple modelling of the three main mechanisms leads to complicated patterns.
- Hydrodynamic Equations without pressure but with nonlocal terms can be at least formally derived.
- Critical thresholds in 1D are obtained for the Euler-type equations. Sharp criteria for alignment but not with attractive-repulsive potentials.
- Sharp Results for thresholds and asymptotic behavior for teh particular case of Newtonian repulsive confined quadratically.
- References:

1. C.-D'Orsogna-Panferov (KRM 2008).
2. C.-Fornasier-Toscani-Vecil (Birkhäuser 2011)
3. C.-Klar-Martin-Tiwari (M3AS 2010).
4. C.-Choi-Tadmor-Tan (M3AS 2016)
5. C.-Choi-Zatorska (M3AS 2016)
6. C.-Choi-Pérez (Book Chapter, Birkhäuser volume edited by Bellomo, Degond & Tadmor)

Conclusions & Open Problems

- Simple modelling of the three main mechanisms leads to complicated patterns.
- Hydrodynamic Equations without pressure but with nonlocal terms can be at least formally derived.
- Critical thresholds in 1D are obtained for the Euler-type equations. Sharp criteria for alignment but not with attractive-repulsive potentials.
- Sharp Results for thresholds and asymptotic behavior for the particular case of Newtonian repulsive confined quadratically.
- References:

1. C.-D'Orsogna-Panferov (KRM 2008).
2. C.-Fornasier-Toscani-Vecil (Birkhäuser 2011)
3. C.-Klar-Martin-Tiwari (M3AS 2010).
4. C.-Choi-Tadmor-Tan (M3AS 2016)
5. C.-Choi-Zatorska (M3AS 2016)
6. C.-Choi-Pérez (Book Chapter, Birkhäuser volume edited by Bellomo, Degond & Tadmor)

Conclusions & Open Problems

- Simple modelling of the three main mechanisms leads to complicated patterns.
- Hydrodynamic Equations without pressure but with nonlocal terms can be at least formally derived.
- Critical thresholds in 1D are obtained for the Euler-type equations. Sharp criteria for alignment but not with attractive-repulsive potentials.
- Sharp Results for thresholds and asymptotic behavior for teh particular case of Newtonian repulsive confined quadratically.
- References:

1. C.-D'Orsogna-Panferov (KRM 2008).
2. C.-Fornasier-Toscani-Vecil (Birkhäuser 2011)
3. C.-Klar-Martin-Tiwari (M3AS 2010).
4. C.-Choi-Tadmor-Tan (M3AS 2016)
5. C.-Choi-Zatorska (M3AS 2016)
6. C.-Choi-Pérez (Book Chapter, Birkhäuser volume edited by Bellomo, Degond & Tadmor)