

Agent Models of 1st and 2nd order: from micro to macro

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Università di Verona, November 2018

Outline

- 1 Modelling
 - Collective Behavior Models
 - Fixed Speed models
- 2 From micro to macro: PDE models
 - Vlasov-like Models
 - Fixed Speed Models as Asymptotic Limits
- 3 Phase Transition for Cucker-Smale
 - Local Cucker-Smale Model
 - Phase Transition driven by Noise
 - Numerical Exploration
- 4 Reduced Hydrodynamics
 - Asymptotic limit
- 5 Conclusions

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Individual Based Models (Particle models)

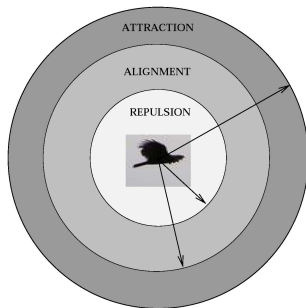
Swarming = Aggregation of agents of similar size and body type generally moving in a coordinated way.

Highly developed social organization: insects (locusts, ants, bees ...), fish, birds, micro-organisms (myxo-bacteria, ...) and artificial robots for unmanned vehicle operation.

Interaction regions between individuals^a

^aAoki, Helmerijk et al., Barbaro, Bimir et al.

- **Repulsion** Region: R_k .
- **Attraction** Region: A_k .
- **Orientation** Region: O_k .



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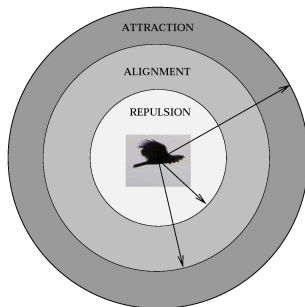
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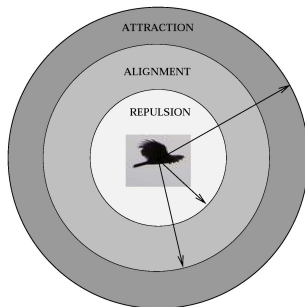
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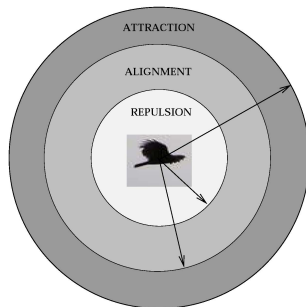
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2nd Order Model: Newton's like equations



D'Orsogna, Bertozzi et al. model (PRL 2006):

$$\begin{cases} \frac{dx_i}{dt} = v_i, \\ m \frac{dv_i}{dt} = (\alpha - \beta |v_i|^2) v_i - \sum_{j \neq i} \nabla U(|x_i - x_j|). \end{cases}$$

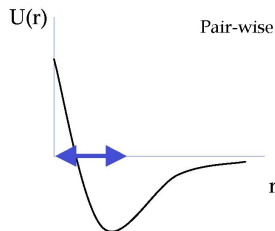
Model assumptions:

- Self-propulsion and friction terms determines an asymptotic speed of $\sqrt{\alpha/\beta}$.
- Attraction/Repulsion modeled by an effective pairwise potential $U(x)$.

$$U(r) = -C_A e^{-r/\ell_A} + C_R e^{-r/\ell_R}.$$

One can also use Bessel functions in 2D and 3D to produce such a potential.

$C = C_R/C_A > 1$, $\ell = \ell_R/\ell_A < 1$ and $C\ell^2 < 1$:



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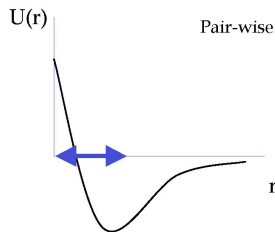
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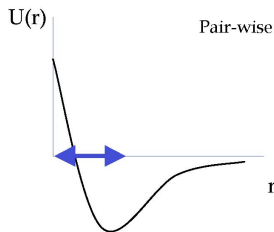
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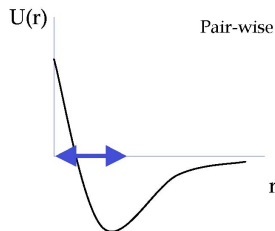
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Velocity consensus model

Cucker-Smale Model (IEEE Automatic Control 2007):

$$\begin{cases} \frac{dx_i}{dt} = v_i, \\ \frac{dv_i}{dt} = \sum_{j=1}^N a_{ij} (v_j - v_i), \end{cases}$$

with the communication rate, $\gamma \geq 0$:

$$a_{ij} = a(|x_i - x_j|) = \frac{1}{(1 + |x_i - x_j|^2)^\gamma}.$$

Asymptotic flocking: $\gamma < 1/2$; Cucker-Smale.

General Proof for $\gamma \leq 1/2$; C.-Fornasier-Rosado-Toscani.

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Vicsek's model

Assume N particles moving at **unit speed**: reorientation & diffusion:

$$\begin{cases} dX_t^i = V_t^i dt, \\ dV_t^i = \sqrt{2D} P(V_t^i) \circ dB_t^i - P(V_t^i) \left(\frac{1}{N} \sum_{j=1}^N K(X_t^i - X_t^j) (V_t^i - V_t^j) \right) dt. \end{cases}$$

Here $P(v)$ is the projection operator on the tangent space at $v/|v|$ to the unit sphere in \mathbb{R}^d , i.e.,

$$P(v) = I - \frac{v \otimes v}{|v|^2}.$$

Noise in the **Stratonovich sense**: imposed by the rigorous construction of the Brownian motion on a manifold. Rigorous derivation: Bolley-Cañizo-C.

Main issue: **phase transition driven by noise D** : Degond-Liu-Frouvelle.

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Convergence of the particle method

Empirical measures: if $x_i, v_i : [0, T) \rightarrow \mathbb{R}^d$, for $i = 1, \dots, N$, is a solution to the ODE system,

$$\left\{ \begin{array}{l} \frac{dx_i}{dt} = v_i, \\ \frac{dv_i}{dt} = \underbrace{(\alpha - \beta |v_i|^2)v_i}_{\text{propulsion-friction}} - \underbrace{\sum_{j \neq i} m_j \nabla U(|x_i - x_j|)}_{\text{attraction-repulsion}} + \underbrace{\sum_{j=1}^N m_j a_{ij} (v_j - v_i)}_{\text{orientation}}. \end{array} \right.$$

then the $f_N : [0, T) \rightarrow \mathcal{P}_1(\mathbb{R}^d)$ given by

$$f_N(t) := \sum_{i=1}^N m_i \delta_{(x_i(t), v_i(t))} \quad \text{with} \quad \sum_{i=1}^N m_i = 1,$$

is expected to be the solution corresponding to initial atomic measures.

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$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = \nabla_v \cdot \left[\underbrace{\left(\int_{\mathbb{R}^{2d}} \frac{v-w}{(1+|x-y|^2)^\gamma} f(y, w, t) dy dw \right)}_{:=\xi(f)(x, v, t)} f(x, v, t) \right]$$

Orientation, Attraction and Repulsion:

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f - \operatorname{div}_v [(\nabla_x U \star \rho) f] = \nabla_v \cdot [\xi(f)(x, v, t) f(x, v, t)].$$

Rigorous proofs of the mean field limit: Cañizo-C.-Rosado (M3AS 2010), Bolley-Cañizo-Rosado (M3AS 2011), C.-Choi-Hauray (Springer Verlag 2012), C.-Choi-Hauray-Salem (to appear in JEMS).

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Short Relaxation towards Cruising Speed

Scaled Vlasov equation in $d = 2, 3$ dimensions:

$$\partial_t f^\varepsilon + v \cdot \nabla_x f^\varepsilon + a^\varepsilon(t, x) \cdot \nabla_v f^\varepsilon + \frac{1}{\varepsilon} \operatorname{div}_v \{f^\varepsilon (\alpha - \beta |v|^2) v\} = 0, \quad (t, x, v) \in \mathbb{R}_+ \times \mathbb{R}^{2d}$$

with $a^\varepsilon(t, \cdot) = -\nabla_x U \star \rho^\varepsilon(t, \cdot) - H \star f^\varepsilon(t, \cdot)$.

This asymptotic limit enforces that particles move at cruising speed $\sqrt{\alpha/\beta}$. If one formally does the expansion

$$f^\varepsilon = f + \varepsilon f^{(1)} + \varepsilon^2 f^{(2)} + \dots$$

we get

$$\begin{aligned} \operatorname{div}_v \{f (\alpha - \beta |v|^2) v\} &= 0 \\ \partial_t f + \operatorname{div}_x (fv) + \operatorname{div}_v (fa(t, x)) + \operatorname{div}_v \{f^{(1)} (\alpha - \beta |v|^2) v\} &= 0, \end{aligned}$$

up to first order.

To eliminate the higher order term we use the invariants of the flow associated to the field $(\alpha - \beta |v|^2) v \cdot \nabla_v$, functions of x and $v/|v|$.

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Vicsek Model as Asymptotic Limit

Bostan-C. (M3AS 2013)

Assume that $U \in C_b^2(\mathbb{R}^d)$, $H(x, v) = h(x)v$ with $h \in C_b^1(\mathbb{R}^d)$ nonnegative, $f^{\text{in}} \in \mathcal{P}_1(\mathbb{R}^d \times \mathbb{R}^d)$, $\text{supp} f^{\text{in}} \subset \{(x, v) : |x| \leq L_0, r_0 \leq |v| \leq R_0\}$.

Then for all $\delta > 0$, **the sequence $(f^\varepsilon)_\varepsilon$** converges towards the measure solution $f(t, x, \omega)$ on $(x, \omega) \in \mathbb{R}^d \times \sqrt{\alpha/\beta}\mathbb{S}$ of the problem

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Remarks:

- Adding noise we get from $\Delta_v f$ to the Laplace-Beltrami operator on the sphere $\Delta_\omega f$. We only know how to perform the formal expansion but not the rigorous limit.
- This formally shows that **the fixed speed limit of the Cucker-Smale's model is the Vicsek's model.**

Vicsek Model as Asymptotic Limit

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The Local Cucker-Smale model with noise

Phase Transition (Barbaro-Cañizo-C.-Degond, SIAM MMS 2016)

- We consider the following kinetic flocking model:

$$\partial_t f + v \nabla_x f = \nabla_v \cdot \left((v - u_f) f - \alpha v (1 - |v|^2) f + D \nabla_v f \right),$$

where

$$u_f(t, x) = \frac{\int v f(t, x, v) dv}{\int f(t, x, v) dv}$$

- The first term is a **Cucker-Smale-like term**, encourages the velocity to align with the mean velocity
- The second term provides self-propulsion and friction, encouraging unit velocities
- The last term captures the influence of noise in the velocity

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The homogeneous problem

- Looking at the spatially homogeneous problem:

$$\partial_t f = \nabla_v \cdot \left((v - u_f) f - \alpha v (1 - |v|^2) f + D \nabla_v f \right)$$

- We have a **gradient flow structure**: write the equation as $\partial_t f = \nabla_v \cdot (f \nabla_v \xi)$ with $\xi = \Phi(v) + W * f + D \log f$

- Confinement in v : $\Phi(v) = \alpha \left(\frac{|v|^4}{4} - \frac{|v|^2}{2} \right)$
 - Interaction potential of the form $W(v) = \frac{|v|^2}{2}$
 - Linear diffusion.
- Our model is continuity equation with velocity field of the form $-\nabla_v \xi$
- Natural entropy** for this equation given by the free energy of the system:

$$\begin{aligned} \mathcal{F}[f] &:= \int_{\mathbb{R}^d} \Phi(v) f(v) dv + \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} W(v-w) f(v) f(w) dw dv + D \int_{\mathbb{R}^d} f(v) \log f(v) dv \\ &= \int_{\mathbb{R}^d} \left(\alpha \frac{|v|^4}{4} + (1 - \alpha) \frac{|v|^2}{2} \right) f(v) dv - \frac{1}{2} |u_f|^2 + D \int_{\mathbb{R}^d} f \log f(v) dv, \end{aligned}$$

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- We consider stationary solutions of the form:

$$f(v) = \frac{1}{Z} \exp \left(\frac{-1}{D} \left[\alpha \frac{|v|^4}{4} + (1 - \alpha) \frac{|v|^2}{2} - u_f \cdot v \right] \right)$$

- We see that in order for the stationary solution to exist, u_f must be a root of the equation:

$$\mathcal{H}(u, D) = \int (v - u)f(v)dv$$

- We prove that, in any dimension¹
 - There is a region of parameter space with **only one** such root, namely $u = 0$
 - There is another region of parameter space with **more than one root**, $u = 0$ and $|u| = C_{\alpha, D} \neq 0$

¹1D case was proven independently in J. Tugaut's *Phase transitions of McKean-Vlasov processes in symmetric and asymmetric multi-wells landscape*, and S. Herrmann and J. Tugaut. *Non-uniqueness of stationary measures for self-stabilizing processes*

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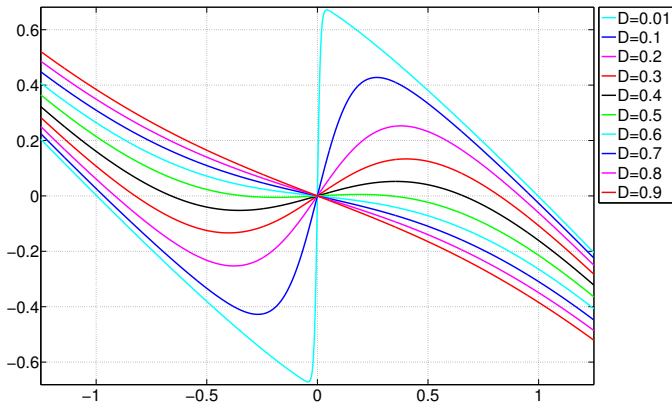
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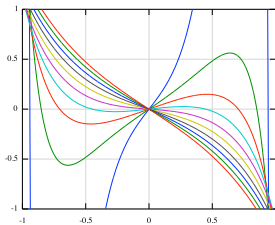
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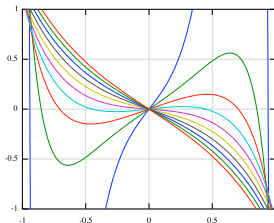


Main idea of our proof



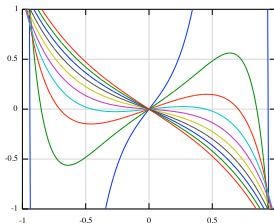
- Our proof hinges Laplace's method and the behavior of $\mathcal{H}(u, D)$ as D varies:
 - For **small** D , we are able to use Laplace's Method to show that there is a nonzero stationary solution
 - For **large** D , $\frac{\partial \mathcal{H}}{\partial u}$ is **negative for all** u .
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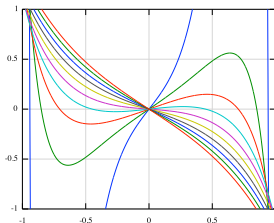
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The case of small D

Find u such that it is a root of $\mathcal{H}(u, D)$, i.e. as $D \rightarrow 0$,

$$u = \frac{\int \exp\left(-\frac{1}{D}P_u(v)\right) v_1 dv}{\int \exp\left(-\frac{1}{D}P_u(v)\right) dv} \quad (1)$$

Laplace's Method tells us that this u must be such that

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- This global minimum is strictly positive
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In order to prove this rigorously, we need to apply an **implicit function theorem from the positive root for $D = 0$** , this needs to compute **next orders in the expansion of Laplace's theorem** and their limits as the noise $D \rightarrow 0$. These expansions are not standard since we need to track carefully the powers of D involved in each term.

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- We show that \mathcal{H} is strictly decreasing in u for $D \rightarrow \infty$
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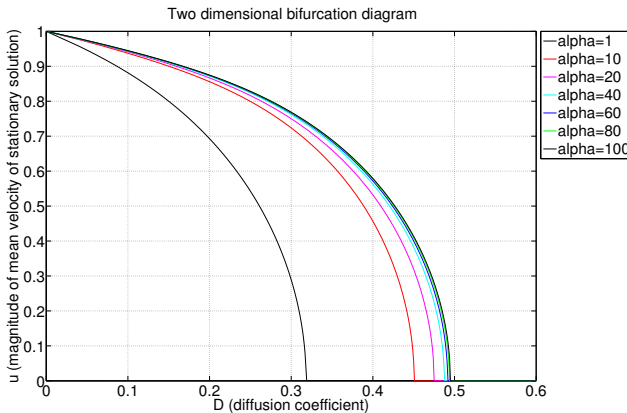
Varying α and D

- We have proven analytically that for small D , there is more than one stationary solutions, while for large D , there is only one
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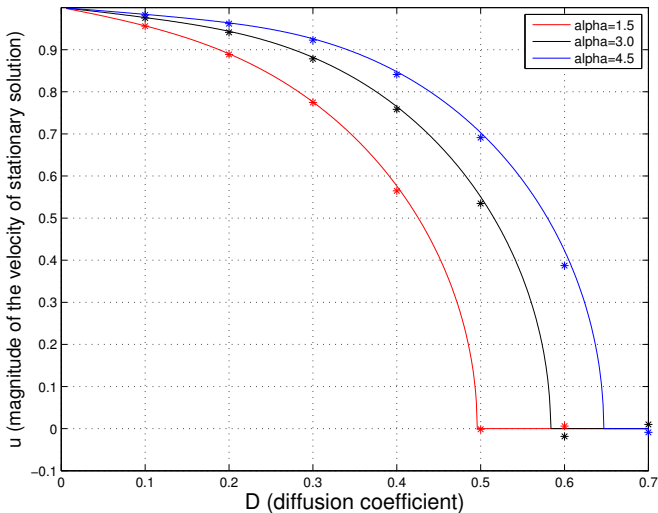
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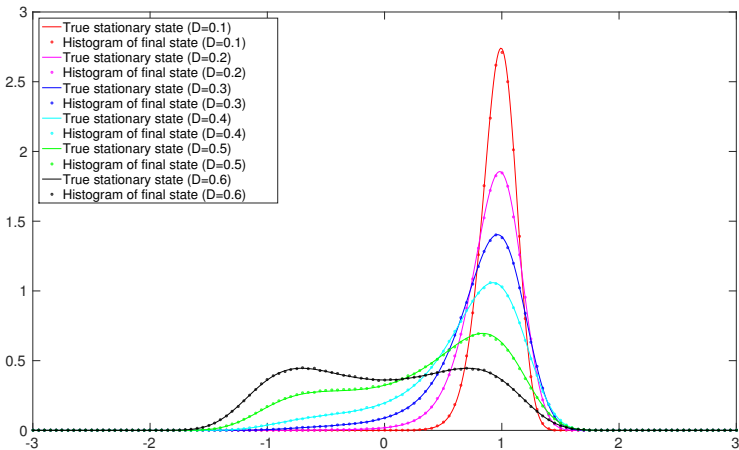
Exploring the limit $\alpha \rightarrow \infty$ in 2D



Stability of the stationary solutions in 1D



Comparing particles to f in 1D



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Hydrodynamics via Asymptotic Limit

Bostan-C. (M3AS 2017)

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for any σ, r such that $\frac{\sigma}{r^2} \in]0, \frac{1}{d}[$, we denote by $l = l\left(\frac{\sigma}{r^2}\right)$ the unique positive solution of $\lambda(l) = \frac{\sigma}{r^2} l$ with

$$\lambda(l) = \frac{\int_0^\pi \cos \theta e^{l \cos \theta} \sin^{d-2} \theta \, d\theta}{\int_0^\pi e^{l \cos \theta} \sin^{d-2} \theta \, d\theta}, \quad l \in \mathbb{R}_+, \quad d \geq 2.$$

Then the limit distribution $f = \lim_{\varepsilon \searrow 0} f^\varepsilon$, is a von Mises-Fisher equilibrium $f = \rho M_{l\Omega}(\omega) \, d\omega$ on $r\mathbb{S}^{d-1}$, where the density $\rho(t, x)$ and the orientation $\Omega(t, x)$ satisfy the macroscopic equations (SOH)

$$\partial_t \rho + \operatorname{div}_x \left(\rho \frac{l\sigma}{r} \Omega \right) = 0, \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$$

$$\partial_t \Omega + k_d r (\Omega \cdot \nabla_x) \Omega + \frac{r}{l} (I_d - \Omega \otimes \Omega) \frac{\nabla_x \rho}{\rho} = 0$$

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Hydrodynamics via Asymptotic Limit

Bostan-C. (M3AS 2017)

Given a solution to

$$\partial_t f^\varepsilon + \operatorname{div}_x(f^\varepsilon v) + \frac{1}{\varepsilon^2} \operatorname{div}_v(f^\varepsilon (\alpha - \beta|v|^2)v) = \frac{1}{\varepsilon} \operatorname{div}_v\{f^\varepsilon (v - u[f^\varepsilon]) + \sigma \nabla_v f^\varepsilon\}$$

for any σ, r such that $\frac{\sigma}{r^2} \in]0, \frac{1}{d}[$, we denote by $l = l\left(\frac{\sigma}{r^2}\right)$ the unique positive solution of $\lambda(l) = \frac{\sigma}{r^2} l$ with

$$\lambda(l) = \frac{\int_0^\pi \cos \theta e^{l \cos \theta} \sin^{d-2} \theta \, d\theta}{\int_0^\pi e^{l \cos \theta} \sin^{d-2} \theta \, d\theta}, \quad l \in \mathbb{R}_+, \quad d \geq 2.$$

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Expansion

The behavior of the family $(f^\varepsilon)_{\varepsilon>0}$, as the parameter ε becomes small, follows by analyzing the formal expansion

$$f^\varepsilon = f + \varepsilon f^{(1)} + \varepsilon^2 f^{(2)} + \dots$$

Plugging the above Ansatz into the kinetic equation, leads to the constraints

$$\operatorname{div}_v \{f(\alpha - \beta|v|^2)v\} = 0$$

$$\operatorname{div}_v \{f^{(1)}(\alpha - \beta|v|^2)v\} = \operatorname{div}_v \{f(v - u[f]) + \sigma \nabla_v f\} := Q(f)$$

and to the time evolution equations

$$\partial_t f + \operatorname{div}_x(fv) + \operatorname{div}_v \{f^{(2)}(\alpha - \beta|v|^2)v\} = \mathcal{L}_f(f^{(1)})$$

with

$$\mathcal{L}_f(f^{(1)}) := \operatorname{div}_v \{f^{(1)}(v - u[f]) + \sigma \nabla_v f^{(1)}\} - \operatorname{div}_v \left\{ f \frac{\int_{\mathbb{R}^d} f^{(1)}(v' - u[f]) \, dv'}{\int_{\mathbb{R}^d} f \, dv'} \right\}$$

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First term

0th-order term in expansion

Assume that $(1 + |v|^2)F \in \mathcal{M}_b^+(\mathbb{R}^d)$. Then F solves $\operatorname{div}_v \{F(\alpha - \beta|v|^2)v\} = 0$ in $\mathcal{D}'(\mathbb{R}^d)$ i.e.,

$$\int_{\mathbb{R}^d} (\alpha - \beta|v|^2)v \cdot \nabla_v \varphi \, dF(v) = 0, \quad \text{for any } \varphi \in C_c^1(\mathbb{R}^d)$$

if and only if $\operatorname{supp} F \subset \{0\} \cup r\mathbb{S}$.

Let $F \in \mathcal{M}_b^+(\mathbb{R}^d)$ be a non negative bounded measure on \mathbb{R}^d . We denote by $\langle F \rangle$ the measure corresponding to the linear application

$$\psi \rightarrow \int_{\mathbb{R}^d} \psi(v) \mathbf{1}_{v=0} F(v) + \int_{\mathbb{R}^d} \psi \left(r \frac{v}{|v|} \right) \mathbf{1}_{v \neq 0} F(v),$$

for all $\psi \in C_c^0(\mathbb{R}^d)$.

Elimination

For any $f \in \mathcal{M}_b^+(\mathbb{R}^d \times \mathbb{R}^d)$ such that $\operatorname{div}_v \{f(\alpha - \beta|v|^2)v\} \in \mathcal{M}_b(\mathbb{R}^d \times \mathbb{R}^d)$, we have $\langle \operatorname{div}_v \{f(\alpha - \beta|v|^2)v\} \rangle = 0$.

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Von Mises Distribution

For any $l \in \mathbb{R}_+$, $\Omega \in \mathbb{S}$, we introduce the **von Mises-Fisher** distribution

$$M_{l\Omega}(\omega) d\omega = \frac{\exp\left(l\Omega \cdot \frac{\omega}{r}\right)}{\int_{r\mathbb{S}^{d-1}} \exp\left(l\Omega \cdot \frac{\omega'}{r}\right) d\omega'} d\omega, \quad \omega \in r\mathbb{S}^{d-1}.$$

Kernel of the averaged collision operator

Let $F \in \mathcal{M}_b^+(\mathbb{R}^d)$ be a non negative bounded measure on \mathbb{R}^d , supported in $r\mathbb{S}^{d-1}$. The following statements are equivalent:

1. $\langle Q(F) \rangle = 0$, that is

$$\int_{v \neq 0} \left\{ -(v - u[F]) \cdot \nabla_v \left[\tilde{\psi} \left(r \frac{v}{|v|} \right) \right] + \sigma \Delta_v \left[\tilde{\psi} \left(r \frac{v}{|v|} \right) \right] \right\} F dv = 0,$$

for all $\tilde{\psi} \in C^2(r\mathbb{S}^{d-1})$.

2. There are $\rho \in \mathbb{R}_+$, $\Omega \in \mathbb{S}$ such that $F = \rho M_{l\Omega} d\omega$ where $l \in \mathbb{R}_+$ satisfies

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- Stability of the symmetric and non-symmetric stationary states as solutions of the homogeneous problem is not analytically known.
- Uniqueness of the non symmetric equilibria except symmetries is open.
- Phase transitions from ordered to disordered state driven by noise in the inhomogeneous case should be explored.
- Reduced Hydrodynamics recovered from the whole space local Cucker-Smale model with noise by asymptotic limits.
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