Kinetic equations with uncertainty: forward and inverse problems

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Autumn School: From Interacting Particle System to Kinetic Equations

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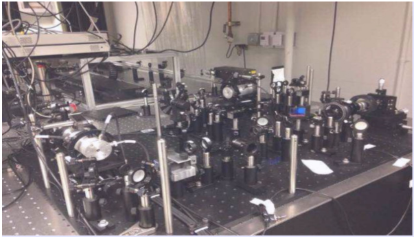
Outline

Source of randomness



Source of uncertainty

- model coefficients
- initial data
- boundary data
- geometry
- ...

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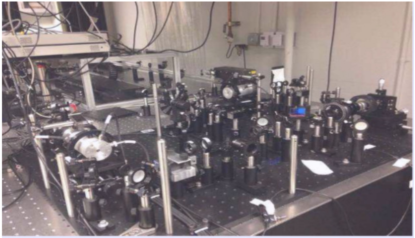

Source of uncertainty

- model coefficients
- initial data
- boundary data
- geometry
- ...

Random input

- Linear kinetic equation: $\partial_t f + \mathsf{T} f = \mathsf{L}(f)$
  - T: transport term
    - $\mathsf{T} = v \cdot \nabla_x$;
    - $\mathsf{T} = v \cdot \nabla_x - \nabla_x V \cdot \nabla_v \,, \quad H(x,v) = \frac{1}{2}|v|^2 + V(x)$;
  - L: collision
    - BGK operator: $\mathsf{L} f = \sigma(x)(\Pi f - f)$;
    - Anisotropic scattering operator:
      $\mathsf{L} f = \int [k(v^* \to v) f(v^*) - k(v \to v^*) f(v)] \mathrm{d}v^*$;
    - Fokker-Planck operator: $\mathsf{L} f = \sigma(x)\,[\nabla_v \cdot (\nabla_v f + v f)]$.
- Nonlinear kinetic equation $\partial_t f + \mathsf{T} f = \mathcal{Q}(f)$
  - Boltzmann $\mathcal{Q}(f) = \int_{R^d} \int_{S^{d-1}} B(v - v_*, \sigma)[f(v') f(v'_*) - f(v) f(v_*)] \mathrm{d}\sigma \mathrm{d}v_*$
  - Landau
    $\mathcal{Q}(f) = \nabla_v \cdot \int_{R^d} A(v - v_*)(f_* \nabla_v f - f \nabla_{v_*} f_*) \mathrm{d}v_*, A(z) = |z|^{\gamma+2} \left(I - \frac{z \otimes z}{|z|^2} \right)$

Random input: our setting

$$\partial_t f + \mathsf{T} f = \mathsf{L}_z f\,, \quad f = f(t,x,v,z) \geq 0$$

- L: collision
  - BGK operator: $\mathsf{L}_z f = \sigma(x,z)(\Pi f - f)$;
  - Anisotropic scattering operator:
    $\mathsf{L}_z f = \int [k_z(v^* \to v) f(v^*) - k_z(v \to v^*) f(v)] \mathrm{d}v^*$;
  - Fokker-Planck operator: $\mathsf{L}_z f = \sigma(x,z)\,[\nabla_v \cdot (\nabla_v f + v f)]$.
- Initial data

$$f(t=0,x,v,z) = f_0(x,v,z) \in L^2(\mathrm{d}x\mathrm{d}v)$$

Uncertainty Quantification

$$\partial_t f + \mathsf{T}(\mathsf{Kn}) f = \mathsf{L}_z(\mathsf{Kn}) f\,, \quad f = f(t,x,v,z) \geq 0$$

Questions

1. How does uncertainty propagate in time?

   $\Longrightarrow$ *Given a smooth dependence of the collision operator and initial data on z, does f enjoy similarly good regularity?*

2. How does uncertainty pass from hyperbolic system to parabolic equation?

   $\Longrightarrow$ *How does regularity depends on Kn?*

Warning: in this lecture, Kn is the scaling parameter

Numerical difficulties & methods

$$\partial_t f + \mathsf{T}(\mathsf{Kn}) f = \mathsf{L}_z(\mathsf{Kn}) f\,, \quad (t, x, v, z) \in [0, \infty) \times \mathbb{R} \times [-1\,, 1] \times \Omega$$

- Multiple scales
  Crouseilles, Degond, Dimarco, Filbet, Gamba, Hauck, Jin, Klar, Li, Lu, Sun, Tang, Pareschi, Qiu, $\cdots$
  - AP method
  - Hybrid method
  - $\cdots$
- Randomenss
  - Monte Carlo sampling
  - generalized Polynomial Chaos(gPC)-based method
    - Stochastic Galerkin: *Babuska-Tempone-Zouraris, Matthies-Keese,* $\cdots$
    - Stochastic Collocation: *Babuska-Nobile-Tempone, Xiu-Hesthaven,* $\cdots$
  - Reduced basis method: *Patera,* $\cdots$
  - $\cdots$
- Both **gPC + AP**

  Jin-Xiu-Zhu, Jin-Liu, Zhu-Jin, Jin-Lu, Shu-Jin, $\cdots$

generalized polynomial chaos (gPC)

$$f(t,x,v,z) \in \text{span}\{f_m(t,x,v)\} \otimes \text{span}\{p_n(z)\}$$

$$f = \sum_{m=0}^{\infty} f_m(t,x,v)p_m(z) \sim \underbrace{\sum_{m=0}^{N} f_m(t,x,v)p_m(z)}_{\mathrm{P_N} f}, \quad f_m = \langle f\,, p_m \rangle_z .$$

$$\partial_t f + \mathsf{T} f = \mathsf{L}_z f$$

- Stochastic Galerkin method

$$f^{\mathrm{G}} = \sum_{m=0}^{N} f_m^{\mathrm{G}}(t,x,v)p_m(z), \qquad \mathrm{P_N}\partial_t f^{\mathrm{G}} + \mathrm{P_N}\mathsf{T}(\mathsf{Kn})f^{\mathrm{G}} = \mathrm{P_N}\mathsf{L}_z(\mathsf{Kn})f^{G}.$$

- Stochastic collocation method
  Solve the deterministic problem on prescribed nodes in the random space, and reconstruct the solution via interpolation

$$f^{\mathrm{C}} = \sum_{m=0}^{N} \tilde{f}_m(t,x,v)p_m(z), \qquad f_m^{C} = \sum_{j=0}^{N} f(z_j)p_m(z_j)\mu_j$$

Goal

$$\textbf{gPC + AP} \stackrel{?}{=} \textbf{stochastic AP}$$

Goal

$$\textbf{gPC + AP} \overset{?}{=} \textbf{stochastic AP}$$

- "probability error"

Goal

gPC + AP $\overset{?}{=}$ stochastic AP

- "probability error"

- [1] Analyticity $\Longrightarrow$  $\Longrightarrow$ exponential convergence of "probability error"

- Uniform analytic regularity: *hypo-coercivity*
  - C. Villani. Hypocoercivity. *Mem. Amer. Math. Soc.*, 2009.
  - F. Hérau and F. Nier. *Archive Ration. Math. Anal.* 2004.
  - C. Mouhot and L.Neumann. *Nonlinearity*, 2006.
  - C. Mouhot. *Comm. Partial Differential Equations*, 2006.
  - J. Dolbeault, C. Mouhot, and C. Schmeiser. *Trans. Amer. Math. Soc.* 2015.
  - E. Daus, A. Jngel, C. Mouhot, and N. Zamponi. *SIAM J. Math. Anal.* 2016.

[1] Babuska-Nobile-Tempone, SIAM J. Num. Anal. 2007 (See Section 4.)

Linear kinetic equation

$$\partial_t f + \mathsf{T} f = \mathsf{L} f\,, \quad f = f(t,x,v) \geq 0$$

- T: transport term
  - $\mathsf{T} = v \cdot \nabla_x$;
  - $\mathsf{T} = v \cdot \nabla_x - \nabla_x V \cdot \nabla_v\,, \quad H(x,v) = \frac{1}{2}|v|^2 + V(x)$;
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  - Fokker-Planck operator: $\mathsf{L} f = \sigma(x)\,[\nabla_v \cdot (\nabla_v f + v f)]$.

Multiple scales

- Diffusive scaling [2]

$$\partial_t f + \frac{1}{\mathsf{Kn}}\mathsf{T}f = \frac{1}{\mathsf{Kn}^2}\mathsf{L}f$$

- "High field" scaling [3]

$$\partial_t f + \frac{1}{\mathsf{Kn}}\mathsf{T}f = \frac{1}{\mathsf{Kn}}\mathsf{L}f$$

[2] Bardos-Santos-Sentis Trans. Amer. Math. Soc. 1984.

[3] Poupaud, Z. Angew Math. Mech. 1992; Cercignani-Gamba-Levermore Appl. Math. Lett. 1997.

Notations

- Null space of L: $\text{Null}\,\mathsf{L} = \text{Span}\{\mathcal{M}(x,v)\} = \{\rho(t,x)\mathcal{M}(x,v)\}$,
- *Local* equilibrium $\mathcal{M}(x,v)$:

$$\iint \mathcal{M}(x,v)\mathrm{d}x\mathrm{d}v = 1,\ \Pi f = \frac{\int f \mathrm{d}v}{\int \mathcal{M}\mathrm{d}v}\mathcal{M}(x,v)$$

- *Global* equilibrium $F(x,v)$:

$$\text{Null}\,\mathsf{L} \cap \text{Null}\,\mathsf{T} = \text{Span}\{F\},\ \iint F(x,v)\mathrm{d}v\mathrm{d}x = 1$$

- Lebesgue measure: $\mathrm{d}\mu = \mathrm{d}\mu(x,v) = \frac{\mathrm{d}x\mathrm{d}v}{F}$
- Hilbert space: $\mathcal{H} = L^2(F^{-1}\mathrm{d}x\mathrm{d}v)$, $\langle f\,, g\rangle = \iint_{\mathbb{R}^d\times\mathbb{R}^d} fg\mathrm{d}\mu$.

Assumptions

- Microscopic coercivity: The operator L is symmetric and

$$-\langle \mathsf{L} f\,, f\rangle \geq \alpha \|(\mathsf{I}-\Pi) f\|^2\,, \quad \text{for all} \quad f \in D(\mathsf{L}), \qquad \alpha > 0\,.$$

- Orthogonality

$$\Pi \mathsf{T} \Pi = 0.$$

- Macroscopic coercivity: The operator T is skew symmetric and

$$\|\mathsf{T}\Pi f\|^2 \geq \beta \|\Pi f\|^2\,, \quad \text{for all} \quad f \in \mathcal{H} \quad \text{s.t.} \quad \Pi f \in D(\mathsf{T}), \qquad \beta > 0.$$

- Denote

$$\mathsf{A} = \left(1 + (\mathsf{T}\Pi)^* (\mathsf{T}\Pi)\right)^{-1} (\mathsf{T}\Pi)^*\,,$$

then $\mathsf{AT}(1-\Pi)$ and AL are both bounded, i.e.,

$$\|\mathsf{AT}(1-\Pi) f\| + \|\mathsf{AL} f\| \leq \gamma \|(1-\Pi) f\|^2\,, \qquad \gamma > 0.$$

Exponential decay of the fluctuation

$$\partial_t f + \mathsf{T} f = \mathsf{L} f$$

Theorem (Dolbeault-Mouhot-Schmeiser, Trans. Amer. Math. Soc., 2015.)

Under the four assumptions, there exists $\lambda(\varepsilon)$ *and* $C(\varepsilon)$ *that are explicitly computable in terms of* α, β, γ *and* ε *such that for any initial datum* $f(0, x, v) \in \mathcal{H}$,

$$\|f\| = \|e^{t(\mathsf{L}-\mathsf{T})} f_0\| \leq C(\varepsilon) e^{-\lambda(\varepsilon) t} \|f_0\| \,,$$

where

$$C(\varepsilon) = \sqrt{\frac{1+\varepsilon}{1-\varepsilon}} \,,$$

and $\varepsilon \in [0, 1)$ *is chosen such that* $\lambda(\varepsilon) > 0$.

Sketch of the proof

- Construct the entropy function:

$$\mathcal{H}(f) = \frac{1}{2}\|f\|^2 + \varepsilon\langle \mathsf{A}f, f\rangle$$

- Decay in time:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}[f] \leq -\frac{2\kappa(\varepsilon)}{1+\varepsilon}\mathcal{H}[f]$$

- Relate to the L^2 norm:

$$\frac{1}{2}(1-\varepsilon)\|f\|^2 \leq \mathcal{H}[f] \leq \frac{1}{2}(1+\varepsilon)\|f\|^2$$

- $\lambda(\varepsilon)$:

$$\begin{aligned}\lambda(\varepsilon) &= \max_{\delta} \frac{\kappa(\varepsilon)}{1+\varepsilon} \\ &= \max_{\delta} \min\left\{ \frac{\alpha - \varepsilon(1+\gamma)\left(1+\frac{1}{2\delta}\right)}{1+\varepsilon}, \quad \frac{\varepsilon}{1+\varepsilon}\left[\frac{\beta}{1+\beta} - (1+\gamma)\frac{\delta}{2}\right]\right\}\end{aligned}$$

$\lambda(\varepsilon)$

$$\|f\| \leq C(\varepsilon) e^{-\lambda(\varepsilon) t} \|f_0\|, \quad C(\varepsilon) = \sqrt{\frac{1+\varepsilon}{1-\varepsilon}}$$

Find *fastest* decay rate:

$$\begin{aligned} \lambda &= \max_{0<\varepsilon<1} \lambda(\varepsilon) \\ &= \max_{\varepsilon,\delta} \min \left\{ \frac{\alpha - \varepsilon(1+\gamma)\left(1+\frac{1}{2\delta}\right)}{1+\varepsilon}, \quad \frac{\varepsilon}{1+\varepsilon}\left[\frac{\beta}{1+\beta} - (1+\gamma)\frac{\delta}{2}\right] \right\}. \end{aligned}$$

- solve the max min problem—find the dependence of λ on α, β and γ;
- how does λ depend on Kn?

Diffusive regime

$$\partial_t f + \frac{1}{\mathsf{Kn}}\mathsf{T}f = \frac{1}{\mathsf{Kn}^2}\mathsf{L}f$$

- In the diffusive regime

$$\alpha_{\mathsf{Kn}} = \frac{\alpha}{\mathsf{Kn}^2}\,, \quad \beta_{\mathsf{Kn}} = \frac{\beta}{\mathsf{Kn}^2}\,, \quad \gamma_{\mathsf{Kn}} = \frac{\gamma}{\mathsf{Kn}}\,,$$

- λ has a lower bound

$$\lambda \geq \frac{\tilde{a}d^2}{(k_0 a + \tilde{a})(k_0 a + c)}\,, \qquad \tilde{a} = \frac{k_0 a^2 c}{k_0 ac + 2dc}\,.$$

$a = \alpha, \quad d = \frac{1+\gamma}{2}, \quad c = \frac{\beta}{1+\beta}.$

Theorem

In the zero limit of Kn, $\lambda_{\mathsf{Kn}} \geq \lambda(\varepsilon_0) \sim \mathcal{O}(1), \quad \varepsilon_0 < 1.$

High field regime

$$\partial_t f + \frac{1}{\mathsf{Kn}}\mathsf{T} f = \frac{1}{\mathsf{Kn}}\mathsf{L} f$$

- In the high field regime

$$\alpha_{\mathsf{Kn}} = \frac{\alpha}{\mathsf{Kn}^2}\,, \quad \beta_{\mathsf{Kn}} = \frac{\beta}{\mathsf{Kn}}\,, \quad \gamma_{\mathsf{Kn}} = \gamma\,,$$

- λ has a lower bound

$$\lambda \geq \lambda(\varepsilon_0) = \begin{cases} \frac{\tilde{a}}{1+\frac{\tilde{a}c}{2d^2}} \frac{\frac{c^2}{d^2}}{2\left[\left(1+\frac{c^2}{2d^2}\right)+\sqrt{1+\frac{c^4}{4d^4}}\right]} & \text{if } \frac{\tilde{a}c}{d^2} \leq 1 \\ \frac{1}{3}\frac{d^2}{(a-d+\frac{1}{2}c)+\sqrt{(a-d-\frac{1}{2}c)^2+d^2}} & \text{if } \frac{\tilde{a}c}{d^2} > 1 \end{cases}, \tilde{a} = \frac{ad}{c+d}$$

$a = \alpha, \quad d = \frac{1+\gamma}{2}, \quad c = \frac{\beta}{1+\beta}.$

Theorem

In the zero limit of Kn, $\lambda_{\mathsf{Kn}} \geq \lambda(\varepsilon_0) \sim \mathcal{O}(1), \quad \varepsilon_0 < 1.$

Analyticity

$$\partial_t f + \mathsf{T} f = \mathsf{L}_z f\,, \quad f(0,x,t,v,z) = f_0(x,v,z)$$

Analyticity

$$\Uparrow$$

- Consider $g_l = \frac{\partial^l}{\partial z^l} f$, then $f(z) = \sum_{l=0}^{\infty} \frac{g_l}{l!}(z-z_0)^l$
- For the above series to converge, need

$$r(z_0) = \frac{1}{\limsup_{l\to\infty} (g_l(z_0)/l!)^{1/l}}\,,$$

to be uniformly bounded from below for all z_0.

Simpler case: $\mathsf{L}_z = \sigma(z,x)\mathsf{L}$

- $g_l = \frac{\partial^l f}{\partial z^l}$ solves:

$$\underbrace{\partial_t g_l + \mathsf{T} g_l = \mathsf{L}_z g_l}_{\text{same as} f} + \underbrace{\sum_{k=0}^{l-1} \frac{l!}{k!(l-k)!} \partial_z^{l-k} \sigma \mathsf{L} g_k}_{\mathcal{S}} \,.$$

- Consider the entropy of g_l: $\mathcal{H}[g_l] = \frac{1}{2}\|g_l\|^2 + \varepsilon_z \langle \mathsf{A} g_l, g_l\rangle$, then

$$\frac{d}{dt}\mathcal{H}[g_l] \leq -2\lambda_z \mathcal{H}[g_l] + \langle \mathcal{S}, g_l\rangle + \varepsilon_z \langle \mathsf{A} g_l, \mathcal{S}\rangle \,.$$

Note that $\langle \mathsf{A} g_l, \mathcal{S}\rangle \leq \|\mathsf{A} g_l\|\|\mathcal{S}\| \leq \frac{1}{2}\|(I-\Pi)g_l\|\|\mathcal{S}\|, \quad \langle \mathcal{S}, g_l\rangle \leq \|\mathcal{S}\|\|g_l\|$

$$\frac{d}{dt}\mathcal{H}[g_l] \leq -2\lambda_z \mathcal{H}[g_l] + (1+\varepsilon_z)\|g_l\|\|\mathcal{S}\|$$

Estimate g_l (1/2) case 1: $\sigma(z,x)$ has an affine dependence on z

Assumption: $\partial_z^l \sigma = 0 \quad \text{for } l > 1, \qquad C_1 = \sup_x |\partial_z \sigma|$.

- estimate of $\mathcal{S}$: $|\mathcal{S}| = \left|\sum_{k=0}^{l-1} \frac{l!}{k!(l-k)!} \partial_z^{l-k} \sigma \mathsf{L} g_k\right| \leq C_1 l |\mathsf{L} g_{l-1}|$
- estimate of $\mathcal{H}[g_l]$:

$$
\begin{aligned}
\frac{d}{dt}\mathcal{H}[g_l] &\leq -2\lambda_z \mathcal{H}[g_l] + (1+\varepsilon_z) C_1 l \|\mathsf{L} g_{l-1}\| \|g_l\| \\
&\leq -2\lambda_z \mathcal{H}[g_l] + (1+\varepsilon_z) C_1 l \|g_{l-1}\| \|g_l\| \\
&\leq -2\lambda_z \mathcal{H}[g_l] + C_1 l (1+\varepsilon_z) \frac{2}{1-\varepsilon_z} \sqrt{\mathcal{H}[g_l]} \sqrt{\mathcal{H}[g_{l-1}]}\,.
\end{aligned}
$$

$$
\Longrightarrow \quad \frac{d}{dt}\sqrt{\mathcal{H}[g_l]} \leq -\lambda_z \sqrt{\mathcal{H}[g_l]} + \tilde{C}_1 l \sqrt{\mathcal{H}[g_{l-1}]}, \qquad \tilde{C}_1 = C_1 C_z^2, \quad C_z = \frac{1+\varepsilon_z}{1-\varepsilon_z}
$$

- Denote $h_l = \sqrt{\mathcal{H}[g_l]} \geq 0$, then

$$
\frac{d}{dt} h_l \leq -\lambda_z h_l + \tilde{C}_1 l h_{l-1}\,.
$$

$$
\Longrightarrow \qquad h_l(t) \leq e^{-\lambda_z t} \sum_{k=0}^{l} \frac{l!}{(l-k)!k!} (\tilde{C}_1 t)^k h_{l-k}(0)\,.
$$

Estimate g_l (2/2) case 1: $\sigma(z,x)$ has an affine dependence on z

Theorem

If the initial data satisfies

$$\|\partial_z^l f_0(z)\| = \|g_l(0)\| \le H^l, \qquad \textit{for all} \;\; l \ge 0$$

then

$$\|g_l\| \le C_z e^{-\lambda_z t}(H + t\tilde{C}_1)^l \,.$$

- Long time behavior: as $t \to \infty$, $\|g_l\| \to 0$
- Convergence radius:

$$r(z_0) = \frac{1}{\limsup_{l\to\infty} \left(g_l(z_0)/l!\right)^{1/l}} = \infty \,.$$

Estimate g_l case 2: $\sigma(z,x)$ has arbitrary dependence on z with $\left|\frac{\partial_z^n \sigma}{n!}\right| \leq C_2$

Theorem

If the initial data satisfies

$$\|\partial_z^l f_0(z)\| = \|g_l(0)\| \leq H, \qquad \textit{for all} \;\; l \geq 0$$

then

$$\left\|\frac{g_l}{l!}\right\| \leq \sqrt{\frac{2}{1-\varepsilon_z}}\frac{H^l}{l!}e^{-\lambda_z t} + \sqrt{\frac{2}{1-\varepsilon_z}}(1+H)^{l+1}\min\{e^{-\lambda_z t}(1+\tilde{C}_2 t)^l, \quad e^{(\tilde{C}_2-\lambda_z)t}2^{l-1}\}.$$

- Long time behavior: as $t \to \infty$, $e^{-\lambda_z t}(1+\tilde{C}_2 t)^l \to 0$
- Convergence radius:

$$r(z_0) = \frac{1}{\limsup_{l\to\infty}\left(e^{(\tilde{C}_2-\lambda_z)t/l}2^{(l-1)/l}(1+H)^{(l+1)/l}\right)} = \frac{1}{2(1+H)}.$$

General form of L_z

$$\partial_t f + \mathsf{T} f = \mathsf{L}_z f, \quad \mathsf{L}_z f(v) = \int [k_z(v^* \to v) f(v^*) - k_z(v \to v^*) f(v)] \mathrm{d}v^*$$

- g_l satisfies:

$$\partial_t g_l + \mathsf{T} g_l = \mathsf{L}_z g_l + \sum_{k=0}^{l-1} \frac{l!}{k!(l-k)!} \mathsf{L}_z^{l-k} g_k$$

$$\mathsf{L}_z^q f = \int \left[\partial_z^q \mathsf{K}_z(v^* \to v) f(v^*) - \partial_z^q \mathsf{K}_z(v \to v^*) f(v)\right] dv$$

- Under the assumption

$$\|\mathsf{L}_z^q f\| \le C_L \|f\|, \quad \forall \ \text{integer } q\,,$$

the analysis is the same as before.

Diffusive scaling

$$\partial_t f + \frac{1}{\mathsf{Kn}} \mathsf{T} f = \frac{1}{\mathsf{Kn}^2} \mathsf{L}_z f$$

- Case 1: $|\partial_z \sigma(x,z)| = C_1$, $\partial_z^l \sigma(x,z) \equiv 0$ for $l \geq 2$, then

Theorem

$$\|g_l\| \leq C e^{-\lambda_{z,\mathsf{Kn}} t} \left(H + t \frac{\tilde{C}_1}{\mathsf{Kn}^2} \right)^l$$

- Case 2: $\left| \frac{\partial_z^l \sigma}{l!} \right| \leq C_2$

Theorem

$$\left\| \frac{g_l}{l!} \right\| \leq \sqrt{\frac{2}{1-\varepsilon_z}} \frac{H^l}{l!} e^{-\lambda_{z,\mathsf{Kn}} t} + \sqrt{\frac{2}{1-\varepsilon_z}} (1+H)^{l+1} \min \left\{ e^{-\lambda_{z,\mathsf{Kn}} t} \left(1 + \frac{\tilde{C}_2}{\mathsf{Kn}^2} t \right)^l , \; e^{\left(\frac{\tilde{C}_2}{\mathsf{Kn}^2} - \lambda_{z,\mathsf{Kn}} \right) t} 2^{l-1} \right\} .$$

High field scaling

$$\partial_t f + \frac{1}{\mathsf{Kn}}\mathsf{T} f = \frac{1}{\mathsf{Kn}}\mathsf{L}_z f$$

- Case 1: $|\partial_z \sigma(x,z)| = C_1$, $\partial_z^l \sigma(x,z) \equiv 0$ for $l \geq 2$, then

Theorem

$$\|g_l\| \leq C e^{-\lambda_{z,\mathsf{Kn}} t}\left(H + t\frac{\tilde{C}_1}{\mathsf{Kn}}\right)^l .$$

- Case 2: $\left|\frac{\partial_z^l \sigma}{l!}\right| \leq C_2$

Theorem

$$\left\|\frac{g_l}{l!}\right\| \leq \sqrt{\frac{2}{1-\varepsilon_z}}\frac{H^l}{l!}e^{-\lambda_{z,\mathsf{Kn}} t} + \sqrt{\frac{2}{1-\varepsilon_z}}(1+H)^{l+1}\min\left\{e^{-\lambda_{z,\mathsf{Kn}} t}\left(1+\frac{\tilde{C}_2}{\mathsf{Kn}}t\right)^l l,\ e^{\left(\frac{\tilde{C}_2}{\mathsf{Kn}}-\lambda_{z,\mathsf{Kn}}\right)t}2^{l-1}\right\}.$$

Other recent results

Linear case

$$\varepsilon \partial_t f + v\partial_x f = \frac{\sigma}{\varepsilon}\mathcal{L}f, \quad \mathcal{L}f = \frac{1}{2}\int_{-1}^{1} f \mathrm{d}v - f, \quad \sigma(x,z) \geq \sigma_{\min} > 0$$

- Hypocoercivity of the linearized kinetic operator: $\frac{1}{\varepsilon}\mathcal{L} - v\partial_x f$
- Jin-Liu-Ma, Research in Math. Sci. 2017
  - Uniform regularity: If $\|D^k f_0(z)\|_{L^\infty} \leq C_0$, $k = 1, 2, \cdots m$, then $\|D^k f\|_\Gamma \leq C$ for $k = 1, 2, \cdots, m$ and $t > 0$.
  - Asymptotic estimate: If further $\sigma \in W^{k,\infty}$, then $\|D^k(\langle f\rangle - f)\|_\Gamma^2 \leq e^{-\sigma_{\min}t/2\varepsilon^2}\|D^k([f_0] - f_0)\|_\Gamma^2 + C'\varepsilon^2$ for any $t \in (0, T]$
  - Uniform convergence: If for some integer $m > 0$, $\|\sigma(z)\|_{H^k} \leq C_\sigma$, $\|D^k f_0\|_\Gamma \leq C_0$, $\|D^k(\partial_x f_0)\|_\Gamma \leq C_x$, for $k = 0, \cdots, m$, then the error of the stochastic Galerkin method is $\|f - f^N\|_\Gamma \leq \frac{C(T)}{N^k}$.
- linear semi-conductor Boltzmann equation: Jin-Liu, Multi. Math. Simu. 2017, Liu KRM 2018.

Nonlinear case

$$\partial_t f + \frac{1}{\varepsilon^\alpha} v \cdot \nabla_x f = \frac{1}{\varepsilon^{1+\alpha}} \mathcal{Q}(f,f), \qquad \alpha = 0\text{: Euler}; \quad \alpha = 1\text{: incompressible N-S}$$

- perturbative setting[4]: $f = \mathcal{M} + \varepsilon\sqrt{\mathcal{M}} h$, $\mathcal{M}(v)$ is the global Maxwellian.

$$\partial_t h + \frac{1}{\varepsilon^\alpha} v \cdot \nabla_x h = \frac{1}{\varepsilon^{\alpha+1}} \mathcal{L}_Q(h) + \frac{1}{\varepsilon^\alpha} \mathcal{Q}(h,h)$$

- $\mathcal{L}$: $\mathcal{N}(\mathcal{L}) = \text{span}\{\psi_1(v), \cdots, \psi_d(v)\}$, $\{\psi_i\}$ are orthonormal
  $\langle \mathcal{L}(h), h\rangle_{L_v^2} \leq -\lambda \|(I - \Pi_{\mathcal{L}})h\|^2_{\Lambda_{\mathcal{Q}}}$ [5]
- General kinetic equation: Liu-Jin Multiscale Model. Simul. 2018
  Assume $\|h(t=0)\|_{H^s_{x,v} L^\infty_z} \leq C_0$, then

$$\alpha = 1: \qquad \|h\|_{H^{s,r}_{x,v} L^\infty_z} \leq C_0 e^{-\tau_s t}, \quad \|h\|_{H^{s,r}_{x,v} H^r_z} \leq C_0 e^{-\tau_s t}$$

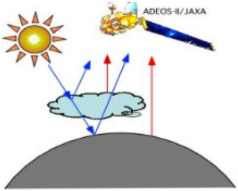
$$\alpha = 0: \qquad \|h\|_{H^{s,r}_{x,v} L^\infty_z} \leq C_0 e^{-\varepsilon\tau_s t}, \quad \|h\|_{H^{s,r}_{x,v} H^r_z} \leq C_0 e^{-\varepsilon\tau_s t}$$

- Vlasov-Poisson-Fokker-Planck system: Jin-Zhu SIAM Math. Anal. 2018

[4] Guo CPAM‘06, Strain-Guo ARMA ‘08

[5] $\lambda_{\mathcal{Q}}$ is problem specific, e.g., see C. Mouhot, Comm. Partial Differential Equations, 2006. for Boltzmann operator ($\|h\|_{\Lambda_{\mathcal{Q}}} = \|h(1+|v|)^{\gamma/2}\|_{L^2}$)

Photon transport



[6] $$\begin{cases} v \cdot \nabla_x f = \int \kappa(x,v,v')f(x,v')\mathrm{d}v' - \sigma(x,v)f \\ f\big|_{\Gamma_-} = \phi(x,v) \end{cases}$$

$$x \in \Omega \subset \mathbb{R}^d, \quad v \in \mathbb{S}^{d-1}$$

$f(t,x,v)$: photon density at location x, with velocity v

$\Gamma_\pm = \{(x,v) : \ x \in \partial\Omega, \pm v \cdot n_x > 0\}$

[6] Dautray-Lions, Springer '93

Diffusive scaling

$$\begin{cases} v \cdot \nabla_x f = \frac{1}{\mathsf{Kn}} \sigma_s \underbrace{(\langle f \rangle - f)}_{\mathcal{L} f} - \mathsf{Kn} \sigma_a f, \quad x \in \Omega, \ v \in \mathbb{S}^{d-1} \\ f\big|_{\Gamma_-} = \phi(x, v) \end{cases}$$

- asymptotic analysis [7] $f = f_0 + \mathsf{Kn} f_1 + \mathsf{Kn}^2 f_2 + \cdots$

$$\mathcal{O}(1/\mathsf{Kn}) \qquad \mathcal{L} f_0 = 0 \qquad \Longrightarrow f_0 = \langle f_0 \rangle = \rho_0$$

$$\mathcal{O}(1) \qquad v \cdot \nabla_x f_0 = \sigma_s \mathcal{L} f_1 \qquad \Longrightarrow f_1 = \mathcal{L}^{-1} \left(\frac{v}{\sigma_s} \cdot \nabla_x f_0 \right) = -\frac{v}{\sigma_s} \cdot \nabla_x \rho_0$$

$$\mathcal{O}(\mathsf{Kn}) \qquad v \cdot \nabla_x f_1 = \sigma_s \mathcal{L} f_2 - \sigma_a f_0 \qquad \Longrightarrow C \nabla_x \cdot \left(\frac{1}{\sigma_s} \nabla_x \rho_0 \right) = \sigma_a \rho_0$$

- boundary data [8] $\rho_0(x) = \xi[\phi], \ x \in \partial\Omega$

[7] Larsen-Keller '74, Bardos-Santos-Sentis '84

[8] Bensoussan-Lions-Papanicolaou '79

Inverse problem set up: transport

$$\begin{cases} v \cdot \nabla_x f = \frac{1}{\mathsf{Kn}} \sigma_s(\langle f \rangle - f) - \mathsf{Kn}\sigma_a f \\ f\big|_{\Gamma_-} = \phi(x,v) \end{cases}$$

- Define Albedo operator: $\mathcal{A}(\sigma_a, \sigma_s): \ \phi(x,v) \ \rightarrow \ \underbrace{f\big|_{\Gamma_+}}_{\text{measurement}}$
- Forward: given σ_s and σ_a, look for $\mathcal{A}$? [9]

$$\sigma_s, \sigma_a \ \rightarrow \ \mathcal{A}: \ \phi(x,v) \text{ to } f\big|_{\Gamma_+}$$

- Inverse: given $\mathcal{A}$, look for σ_s and σ_a? [10]

$$\sigma_s, \sigma_a \ \leftarrow \ \mathcal{A}: \ \phi(x,v) \text{ to } f\big|_{\Gamma_+}$$

[9] Dautray-Lions '93

[10] Bal, Stefenov, Uhlmann, Ren, Hielscher, Arridge, Tamasan, Chung, etc.

Inverse problem set up: diffusion

$$\begin{cases} C\nabla_x \cdot \left(\frac{1}{\sigma_s}\nabla_x \rho\right) = \sigma_a \rho \\ \rho\big|_{\partial\Omega} = \psi(x) \end{cases}$$

- Define Dirichlet to Neumann map: $\mathcal{A}: \ \psi(x) \to \frac{\partial \rho}{\partial n}\big|_{\partial\Omega}$
- Forward: given σ_s and σ_a, look for $\mathcal{A}$?

$$\sigma_s, \sigma_a \ \to \ \mathcal{A}: \ \psi(x) \text{ to } \frac{\partial \rho}{\partial n}\Big|_{\partial\Omega}$$

- Inverse: given $\mathcal{A}$, look for σ_s and σ_a? [11]

$$\sigma_s, \sigma_a \ \leftarrow \ \mathcal{A}: \ \psi(x) \text{ to } \frac{\partial \rho}{\partial n}\Big|_{\partial\Omega}$$

[11] Uhlmann, ...

Connections?

Transport

$$\begin{cases} v \cdot \nabla_x f = \frac{1}{\mathsf{Kn}} \sigma_s^{\mathrm{T}} (\langle f \rangle - f) - \mathsf{Kn} \sigma_a^{\mathrm{T}} f \\ f\big|_{\Gamma_-} = \phi(x, v) \end{cases}$$

$$\sigma_s^{\mathrm{T}}, \sigma_a^{\mathrm{T}} \leftarrow \phi(x, v) \text{ to } f\big|_{\Gamma_+}$$

well-posed [12]

Diffusion

$$\begin{cases} C\nabla_x \cdot \left(\frac{1}{\sigma_s^{\mathrm{D}}} \nabla_x \rho \right) = \sigma_a^{\mathrm{D}} \rho \\ \rho\big|_{\partial\Omega} = \psi(x) \end{cases}$$

$$\sigma_s^{\mathrm{D}}, \sigma_a^{\mathrm{D}} \leftarrow \psi(x) \text{ to } \frac{\partial \rho}{\partial n}|_{\partial\Omega}$$

Calderón-like problem: ill-posed [13]

1. When $(\sigma_s^{\mathrm{D}}, \sigma_a^{\mathrm{D}})$ is a good approximation of $(\sigma_s^{\mathrm{T}}, \sigma_a^{\mathrm{T}})$?
2. What is lost when diffusion approximation is performed? *well-posedness*
3. Will such approximation affect the *stability* in the inverse problem?

[12] Choulli-Stefanov '96, '98, Stefanov-Tamasan '09

[13] Greenleaf-Lassas-Uhlmann, '03, Uhlmann '09, Arridge-Lionheart, '98

Connections?

Transport

$$\begin{cases} v \cdot \nabla_x f = \frac{1}{\mathsf{Kn}} \sigma_s^{\mathrm{T}} (\langle f \rangle - f) - \mathsf{Kn} \sigma_a^{\mathrm{T}} f \\ f\big|_{\Gamma_-} = \phi(x, v) \end{cases}$$

$$\sigma_s^{\mathrm{T}}, \sigma_a^{\mathrm{T}} \leftarrow \phi(x,v) \text{ to } f\big|_{\Gamma_+}$$

well-posed [14]

Diffusion

$$\begin{cases} C \nabla_x \cdot \left(\frac{1}{\sigma_s^{\mathrm{D}}} \nabla_x \rho \right) = \sigma_a^{\mathrm{D}} \rho \\ \rho\big|_{\partial\Omega} = \psi(x) \end{cases}$$

$$\sigma_s^{\mathrm{D}}, \sigma_a^{\mathrm{D}} \leftarrow \psi(x) \text{ to } \frac{\partial \rho}{\partial n}\Big|_{\partial\Omega}$$

Calderón-like problem: ill-posed [15]

Our goal (dependence on Kn)

1. 1D: study injectivity + stability
2. Higher D: assume *injectivity*, study stability

[14] Choulli-Stefanov '96, '98, Stefanov-Tamasan '09

[15] Greenleaf-Lassas-Uhlmann, '03, Uhlmann '09, Arridge-Lionheart, '98

Recover σ_s

Original

$$\begin{cases} v \cdot \nabla_x f = \frac{1}{\mathsf{Kn}} \sigma_s \mathcal{L} f - \mathsf{Kn} \sigma_a f \\ f\big|_{\Gamma_-} = \phi \end{cases}$$

Linearization

$$\begin{cases} v \cdot \nabla_x f_0 = \frac{1}{\mathsf{Kn}} \sigma_{s0} \mathcal{L}(f_0) - \mathsf{Kn} \sigma_a f_0 \\ f_0\big|_{\Gamma_-} = \phi \end{cases}$$

- Take the difference: $\tilde{f} = f - f_0$ and $\tilde{\sigma}_s = \sigma_s - \sigma_{s0}$

$$\begin{cases} v \cdot \nabla_x \tilde{f} = \frac{1}{\mathsf{Kn}} \sigma_{s0} \mathcal{L} \tilde{f} + \frac{1}{\mathsf{Kn}} \tilde{\sigma}_s \mathcal{L} f_0 - \mathsf{Kn} \sigma_a \tilde{f}\,, \\ \tilde{f}|_{\Gamma_-} = 0\,. \end{cases}$$

- The adjoint problem:

$$\begin{cases} -v \cdot \nabla_x g = \frac{1}{\mathsf{Kn}} \sigma_{s0} \mathcal{L} g - \mathsf{Kn} \sigma_a g\,, \\ g|_{\Gamma_+} = \delta_y(x)\,. \end{cases}$$

Recover σ_s

Original

$$\begin{cases} v \cdot \nabla_x f = \frac{1}{\mathsf{Kn}} \sigma_s \mathcal{L} f - \mathsf{Kn} \sigma_a f \\ f\big|_{\Gamma_-} = \phi \end{cases}$$

Linearization

$$\begin{cases} v \cdot \nabla_x f_0 = \frac{1}{\mathsf{Kn}} \sigma_{s0} \mathcal{L}(f_0) - \mathsf{Kn} \sigma_a f_0 \\ f_0\big|_{\Gamma_-} = \phi \end{cases}$$

- Take the difference: $\tilde{f} = f - f_0$ and $\tilde{\sigma}_s = \sigma_s - \sigma_{s0}$

$$\begin{cases} v \cdot \nabla_x \tilde{f} = \frac{1}{\mathsf{Kn}} \sigma_{s0} \mathcal{L} \tilde{f} + \frac{1}{\mathsf{Kn}} \tilde{\sigma}_s \mathcal{L} f_0 - \mathsf{Kn} \sigma_a \tilde{f} \,, \\ \tilde{f}|_{\Gamma_-} = 0 \,. \end{cases} \qquad \times \; g$$

- The adjoint problem:

$$\begin{cases} -v \cdot \nabla_x g = \frac{1}{\mathsf{Kn}} \sigma_{s0} \mathcal{L} g - \mathsf{Kn} \sigma_a g \,, \\ g|_{\Gamma_+} = \delta_y(x) \,. \end{cases} \qquad \times \; \tilde{f}$$

- $$\underbrace{\int v \cdot n \tilde{f} g|_{\Gamma_+} \mathrm{d}v \mathrm{d}x}_{\text{measurement - computed data}} = \int_\Omega \tilde{\sigma}_s \underbrace{\frac{1}{\mathsf{Kn}} \int g \mathcal{L} f_0 \mathrm{d}v}_{\gamma_{\mathsf{Kn}}(x;\delta_y,\phi)} \mathrm{d}x$$

- $\boxed{\int_\Omega \gamma_{\mathsf{Kn}}(x;\delta_y,\phi) \tilde{\sigma}_s(x) \mathrm{d}x = b(\delta_y,\phi)}$ *1st-type Fredholm integral*

Recover σ_s: *non-injectivity in 1D*

$$\boxed{\langle \gamma_{\mathsf{Kn}}, \tilde{\sigma}_s \rangle_{L^2(\mathrm{d}x)} = b(\delta_y, \phi), \quad \gamma_{\mathsf{Kn}} = \frac{1}{\mathsf{Kn}} \int \mathcal{L} f_0 g \mathrm{d}v}$$

Q : When could $\tilde{\sigma}_s(x)$ be uniquely determined?

A : Suppose $\tilde{\sigma}_s \in L^2$, need $\mathrm{span}\{\gamma_{\mathsf{Kn}}(x; \delta_y, \phi)\} = L^2$.
Otherwise $\tilde{\sigma}_s + \mathrm{span}\{\gamma\}^\perp$ are all solutions. Non-uniqueness.

Theorem (Chen-Li-W.)

For arbitrary inflow data ϕ and Dirac delta function δ_y

- *if $\sigma_a \equiv 0$, γ_{Kn} is a constant independent of x;*
- *if $\sigma_a > 0$, $\frac{d}{dx}\gamma_{\mathsf{Kn}} \to 0$ as $\mathsf{Kn} \to 0$.*

Therefore, when $\mathsf{Kn} \to 0$, solving linear system $\langle \gamma_{\mathsf{Kn}}, \tilde{\sigma}_s \rangle_{L^2(\mathrm{d}x)} = b$ is non-injective.

Recover σ_s: *ill-conditioning in higher-D*

$$\boxed{\langle \gamma_{\mathsf{Kn}}, \tilde{\sigma}_s \rangle_{L^2(\mathrm{d}x)} = b(\delta_y, \phi), \quad \gamma_{\mathsf{Kn}} = \frac{1}{\mathsf{Kn}} \int \mathcal{L} f_0 g \mathrm{d}v}$$

Theorem (Chen-Li-W.)

Define the distinguishability coefficient

$$\kappa_s := \sup_{\sigma_s \in \Gamma_\delta} \frac{\|\sigma_s - \tilde{\sigma}_s\|_{L^\infty(\mathrm{d}x)}}{\|\tilde{\sigma}_s\|_{L^\infty(\mathrm{d}x)}},$$

where $\Gamma_\delta = \{\sigma : \sup_{\substack{\forall \|\phi\|_{L^\infty(\Gamma_-)} \leq 1, \\ \forall y \in \partial\Omega}} |\langle \gamma_{\mathsf{Kn}}, \sigma \rangle_{L^2(\mathrm{d}x)} - b(\delta_y, \phi_d)| \leq \delta\}$. *Then*

$$\kappa_s \geq \mathcal{O}\left(\frac{\delta}{\mathsf{Kn}}\right), \quad \textit{when} \quad \mathsf{Kn} \ll 1.$$

- small δ leads to better distinguishability;
- small Kn leads to worse distinguishability;

Recover σ_a

Original

$$\begin{cases} v \cdot \nabla_x f = \frac{1}{\mathsf{Kn}}\mathcal{L}f - \mathsf{Kn}\sigma_a f \\ f\big|_{\Gamma_-} = \phi \end{cases}$$

Linearization

$$\begin{cases} v \cdot \nabla_x f_0 = \frac{1}{\mathsf{Kn}}\mathcal{L}(f_0) - \mathsf{Kn}\sigma_{a_0} f_0 \\ f_0\big|_{\Gamma_-} = \phi \end{cases}$$

- Take the difference: $\tilde{f} = f - f_0$ and $\tilde{\sigma}_a = \sigma_a - \sigma_{a0}$

$$\begin{cases} v \cdot \nabla_x \tilde{f} = \frac{1}{\mathsf{Kn}}\mathcal{L}\tilde{f} + \frac{1}{\mathsf{Kn}}\tilde{\sigma}_s\mathcal{L}f_0 - \mathsf{Kn}\sigma_a \tilde{f}\,, \\ \tilde{f}|_{\Gamma_-} = 0\,. \end{cases}$$

- The adjoint problem:

$$\begin{cases} -v \cdot \nabla_x g = \frac{1}{\mathsf{Kn}}\mathcal{L}g - \mathsf{Kn}\sigma_a g\,, \\ g|_{\Gamma_+} = \delta_y(x)\,. \end{cases}$$

Recover σ_a

Original

$$\begin{cases} v \cdot \nabla_x f = \frac{1}{\mathsf{Kn}}\mathcal{L}f - \mathsf{Kn}\sigma_a f \\ f\big|_{\Gamma_-} = \phi \end{cases}$$

Linearization

$$\begin{cases} v \cdot \nabla_x f_0 = \frac{1}{\mathsf{Kn}}\mathcal{L}(f_0) - \mathsf{Kn}\sigma_{a_0} f_0 \\ f_0\big|_{\Gamma_-} = \phi \end{cases}$$

- Take the difference: $\tilde{f} = f - f_0$ and $\tilde{\sigma}_a = \sigma_a - \sigma_{a0}$

$$\begin{cases} v \cdot \nabla_x \tilde{f} = \frac{1}{\mathsf{Kn}}\mathcal{L}\tilde{f} + \frac{1}{\mathsf{Kn}}\tilde{\sigma}_s\mathcal{L}f_0 - \mathsf{Kn}\sigma_a\tilde{f}\,, \\ \tilde{f}|_{\Gamma_-} = 0\,. \end{cases} \qquad \times\ g$$

- The adjoint problem:

$$\begin{cases} -v \cdot \nabla_x g = \frac{1}{\mathsf{Kn}}\mathcal{L}g - \mathsf{Kn}\sigma_a g\,, \\ g|_{\Gamma_+} = \delta_y(x)\,. \end{cases} \qquad \times\ \tilde{f}$$

- $$\underbrace{\int v \cdot n\tilde{f}g|_{\Gamma_+}\mathrm{d}v\mathrm{d}x}_{\text{measurement - computed data}} = \int_\Omega \tilde{\sigma}_a \underbrace{\mathsf{Kn}\int f_0 g\mathrm{d}v}_{\gamma_{\mathsf{Kn}}(x;\delta_y,\phi)}\mathrm{d}x$$

- $\boxed{\int_\Omega \gamma_{\mathsf{Kn}}(x;\delta_y,\phi)\tilde{\sigma}_a(x)\mathrm{d}x = b(\delta_y,\phi)}$ *1st-type Fredholm integral*

Recover σ_a: ill-conditioning in *continuous* level

$$\boxed{\langle \gamma_{\mathsf{Kn}}, \tilde{\sigma}_a \rangle_{L^2(\mathrm{d}x)} = b(\delta_y, \phi), \quad \gamma_{\mathsf{Kn}} = \mathsf{Kn} \int f_0 g \mathrm{d}v}$$

Theorem (Chen-Li-W.)

Define the distinguishability coefficient

$$\kappa_a := \sup_{\sigma_a \in \Gamma_\delta} \frac{\|\sigma_a - \tilde{\sigma}_a\|_{L^\infty(\mathrm{d}x)}}{\|\tilde{\sigma}_a\|_{L^\infty(\mathrm{d}x)}},$$

where $\Gamma_\delta = \{\sigma : \sup_{\substack{\forall \|\phi\|_{L^\infty(\Gamma_-)} \le 1, \\ \forall y \in \partial\Omega}} |\langle \gamma_{\mathsf{Kn}}, \sigma \rangle_{L^2(\mathrm{d}x)} - b(\delta_y, \phi_d)| \le \delta\}$. *Then*

$$\kappa_a \ge \mathcal{O}\left(\frac{\delta}{\mathsf{Kn}^2}\right), \quad \textit{when} \quad \mathsf{Kn} \ll 1.$$

Recover σ_a: ill-conditioning in *discrete* level

$$\boxed{\langle \gamma_{\mathsf{Kn}}, \tilde{\sigma}_a \rangle_{L^2(\mathrm{d}x)} = b(\delta_y, \phi), \quad \gamma_{\mathsf{Kn}} = \mathsf{Kn} \int f_0 g \mathrm{d}v}$$

Discrete level:

$$\underbrace{\sum_{i=1}^{N_x} \gamma_{\mathsf{Kn}}(x_i; y_k, \phi_d) \tilde{\sigma}_a(x_i) w_i}_{\mathsf{A}\tilde{\sigma}_{\mathbf{a}}} = \underbrace{b(y_k, \phi_d)}_{\mathbf{b}}, \ \forall d = 1, \cdots, N_\phi, \quad k = 1, \cdots, N_y$$

Theorem (Chen-Li-W.)

The condition number of matrix $\mathsf{A}^T\mathsf{A}$ *scales as*

$$cond(\mathsf{A}^T\mathsf{A}) \sim \mathcal{O}\left(\frac{1}{\mathsf{Kn}}\right).$$

Moreover, in 1D, A *is approximately low rank, in the sense that it only has no more than 3 singular values of size* $\mathcal{O}(\mathsf{Kn})$*, and all the rest are of size* $\mathcal{O}(\mathsf{Kn}^{3/2})$*.*

Outlook

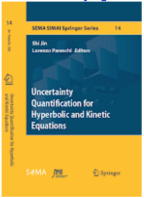
- Forward problem
  - collisionless kinetic equation: Landau damping
    - Z. Ding and S. Jin, Random regularity of a nonlinear Landau Damping solution for the Vlasov-Poisson equations with random inputs, *Int'l J. Uncertainty Quantification*, to appear.
    - R. Shu and S. Jin, A study of Landau damping with random initial inputs, *J. Diff. Eqn.*, to appear
  - high dimensional randomness: randomized/adaptive solver
    - Multilevel Monte Carlo: Multi-scale control variate methods for uncertainty quantification in kinetic equations, G. Dimarco, L. Pareschi, *submitted.*
- Inverse problem
  - nonlinearity
  - efficient algorithms

References

Review articles

- A book

Uncertainty Quantification for Hyperbolic and Kinetic Equations



Shi Jin, Lorenzo Pareschi (Eds.)
SEMA SIMAI Springer Series
277 pages, 2018

This book explores recent advances in uncertainty quantification for hyperbolic, kinetic, and related problems. The contributions address a range of different aspects, including: polynomial chaos expansions, perturbation methods, multi-level Monte Carlo methods, importance sampling, and moment methods. The interest in these topics is rapidly growing, as their applications have now expanded to many areas in engineering, physics, biology and the social sciences. Accordingly, the book provides the scientific community with a topical overview of the latest research efforts.

- J. Hu and S. Jin, Uncertainty Quantification for Kinetic Equations. *in "Uncertainty Quantification for Kinetic and Hyperbolic Equations,"* pp. 193-229, SEMA-SIMAI Springer Series, ed. S. Jin and L. Pareschi, Springer, 2017.

- S. Jin, Mathematical Analysis and Numerical Methods for Multiscale Kinetic Equations with Uncertainties. *Proceedings of The International Congress of Mathematicians*, Rio de Janeiro, Vol. 3, 3595-3624, 2018.

- Shi Jin's web: http://www.math.wisc.edu/ jin/research.html